Exercise Set 4 : 15 March 2018
Calcul Quantique

Exercise 1 *1-qubit non-classical gate*

Consider a gate $G = \begin{pmatrix} \frac{1}{2} + \frac{i}{2} & \frac{1}{2} - \frac{i}{2} \\ \frac{1}{2} - \frac{i}{2} & \frac{1}{2} + \frac{i}{2} \end{pmatrix}$.

(a) Show that G^2 is a real matrix.

(b) For $|v\rangle = |0\rangle, |1\rangle$ and $\alpha|0\rangle + \beta|1\rangle$, compute $G|v\rangle$ and $G^2|v\rangle$.

(c) The gate G is a purely quantum logic gate without any classical equivalent. However, G^2 is equivalent to a classical logic gate. What classical logic gate is it ?

Exercise 2 *Superposition of exponentially many component states*

Suppose we have n qubits, all prepared in state $|0\rangle$. We want to obtain an equal superposition of 2^n component states, given by

$$\frac{1}{\sqrt{2^n}} \sum_{b_1, b_2, \ldots, b_m \in \{0,1\}^2} |b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_n\rangle \,. \tag{1}$$

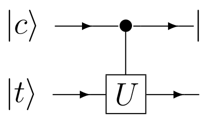
For example, when $n = 2$, we want to obtain

$$\frac{1}{2}|0\rangle \otimes |0\rangle + \frac{1}{2}|0\rangle \otimes |1\rangle + \frac{1}{2}|1\rangle \otimes |0\rangle + \frac{1}{2}|1\rangle \otimes |1\rangle \,.$$

Can you create a circuit that outputs (1) ? Start thinking with $n = 1, 2$.

Exercise 3 *SWAP · controlled-U · SWAP*

Let $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$. The controlled-$U$ gate is given by the following circuit :



$$U^c|t\rangle = \begin{cases} |t\rangle \text{ if } c = 0 \\ U|t\rangle \text{ if } c = 1 \end{cases}$$

(a) Show that the controlled-U has the following matrix representation :

$$\text{controlled-}U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{11} & U_{12} \\ 0 & 0 & U_{21} & U_{22} \end{pmatrix}$$

(b) What is the circuit for SWAP $\cdot$ controlled-U $\cdot$ SWAP ?

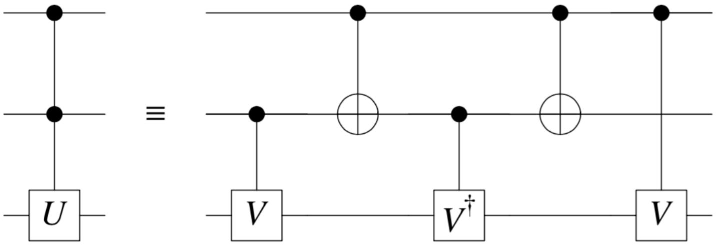
(c) Check that SWAP $\cdot$ controlled-U $\cdot$ SWAP has the following matrix representation :

$$\text{SWAP} \cdot \text{controlled-}U \cdot \text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & U_{11} & 0 & U_{12} \\ 0 & 0 & 1 & 0 \\ 0 & U_{21} & 0 & U_{22} \end{pmatrix}$$

Exercise 4 *Contolled-controlled-U*

In the last exercise, we have seen the construction of multi-controlled-U gate with the Toffoli gates and a simple controlled-U gate. This time, we are going to see a different construction for controlled-controlled-U gate.

Let V be any quantum gate such that $V^2 = U$. Prove the following circuit identity.



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