Midterm

Nom: ........................................ Prénom: ................................. Section: .....................

• Vous pouvez répondre aux questions en Français ou en Anglais.
• Ecrivez votre nom sur chaque feuille double, rendez la donnée et tous les brouillons SVP.
• Durée: 8h15-10h00.
• Les formules suivantes de trigonométrie peuvent être utiles:

\[
\cos^2 \alpha + \sin^2 \alpha = 1, \quad \cos^2 \alpha - \sin^2 \alpha = \cos(2\alpha), \quad 2 \cos \alpha \sin \alpha = \sin(2\alpha).
\]

• Vous avez le droit de faire les calculs en notation de Dirac ou en composantes si vous préférez (on prend tout ce qui est juste).

Exercise 1 Interferometer with imperfect semi-transparent mirrors

We consider the interferometer constituted of two semi-transparent and two reflecting mirrors. We assume the Hilbert space of the photon is \( \mathcal{H} = \mathbb{C}^2 = \{ |\Psi\rangle = \alpha |h\rangle + \beta |v\rangle, |\alpha|^2 + |\beta|^2 = 1 \} \) where \( |h\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( |v\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). The incoming photon on the left is in state \( |h\rangle \).

The semi-transparent mirrors have “imperfections” and modeled by the matrix

\[
S = \cos \gamma |h\rangle \langle h| + \sin \gamma |h\rangle \langle v| + \sin \gamma |v\rangle \langle h| - \cos \gamma |v\rangle \langle v| = \begin{pmatrix} \cos \gamma & \sin \gamma \\ \sin \gamma & -\cos \gamma \end{pmatrix}
\]

The perfectly reflecting mirrors are modeled by

\[
R = |h\rangle \langle v| + |v\rangle \langle h| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

(a) Compute the state after the second semi-transparent mirrors (just before the detectors).
(b) Compute the probabilities of detecting the photon in the detectors $P(D_1)$ and $P(D_2)$.

c) What are the probabilities for $\gamma = \frac{\pi}{4}$? For $\gamma = \frac{\pi}{8}$? (express the probabilities as simple rational fractions using $\cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$).

d) Prove that $S$ and $R$ are unitary.

**Exercise 2 Quantum key distribution when Alice has an imperfect encoder**

Consider the BB84 protocol with the following modification.

- Alice generates two random sequences $x_1 \ldots x_N$ with $P(x_i = 0) = P(x_i = 1) = 1/2$ and $e_1 \ldots e_N$ with $P(e_i = 0) = P(e_i = 1) = 1/2$.

- For $e_i = 0$ she encodes the $i$-th qubit in the state $|x_i\rangle$. For $e_i = 1$ she encodes the $i$-th qubit in the state $U|x_i\rangle$, where

$$U = \cos \frac{\pi}{8} |0\rangle \langle 0| + \sin \frac{\pi}{8} |0\rangle \langle 1| + \sin \frac{\pi}{8} |1\rangle \langle 0| - \cos \frac{\pi}{8} |1\rangle \langle 1| .$$

- Bob generates a random sequence $d_1 \ldots d_N$ with $P(d_i = 0) = P(d_i = 1) = 1/2$.
  - If $d_i = 0$, he measures the received qubit in the basis $\{|0\rangle, |1\rangle\}$ and registers $y_i = 0$ (respectively $y_i = 1$) if the outcome is $|0\rangle$ (respectively $|1\rangle$).
  - If $d_i = 1$, he measures the received qubit in the basis $\left\{ \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle), \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \right\}$ and registers $y_i = 0$ (respectively $y_i = 1$) if the outcome is $\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$ (respectively $\frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$).

(a) Compute the probabilities $P(x_i = y_i|e_i = d_i = 0)$ and $P(x_i = y_i|e_i = d_i = 1)$.

*Hint:* For any event $A$ you have:

$$P(x_i = y_i|A) = P(x_i = y_i|A, x_i = 0)P(x_i = 0) + P(x_i = y_i = 1|A, x_i = 1)P(x_i = 1)$$

(b) Deduce the probability $P(x_i = y_i|e_i = d_i)$. How does it compare to the corresponding probability in the perfect BB84 protocol?

**Exercise 3 Entanglement for two quantum bits**

The state of two quantum bits has a general form

$$|\Psi\rangle = a_{00}|0\rangle \otimes |0\rangle + a_{01}|0\rangle \otimes |1\rangle + a_{10}|1\rangle \otimes |0\rangle + a_{11}|1\rangle \otimes |1\rangle,$$

where $\{|0\rangle, |1\rangle\}$ is the canonical orthonormal basis of $\mathbb{C}^2$.

(a) Show that $|\Psi\rangle$ is a product state in the sense that it admits a “product form” $(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$ if and only if $\det A = 0$, where $A$ is the matrix

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}.$$

*Hint:* Recall that $\det A = a_{00}a_{11} - a_{10}a_{01}$.

(b) Using part (a), show that

$$|\Psi_1\rangle = \frac{1}{\sqrt{3}} \left( |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + i |1\rangle \otimes |0\rangle \right)$$

is entangled (in french "intriquer").