Homework 7: 3 November 2016
Traitement Quantique de l'Information

Exercise 1 *Bell states*

The purpose of this exercise is to get you familiar with calculations involving Bell states: the calculations of the first three questions are in Dirac notation.

1) Show that the four Bell states introduced in class form an orthonormal basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$.

2) Show that the state $|B_{00}\rangle$ (or take any other Bell states you like) is entangled. This means that there does not exist $|\phi_1\rangle, |\phi_2\rangle \in \mathbb{C}^2$ such that $|B_{00}\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \in \mathbb{C}^2$.

3) Let $|\gamma\rangle = \cos\gamma\,|0\rangle + \sin\gamma\,|1\rangle$ and $|\gamma_\perp\rangle = -\sin\gamma\,|0\rangle + \cos\gamma\,|1\rangle$. Show the identity (for all angle of polarization γ)

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}(|\gamma\rangle \otimes |\gamma\rangle + |\gamma_\perp\rangle) \otimes |\gamma_\perp\rangle \,.$$

4) Represent the four Bell states in coordinates. Use the canonical representation $|0\rangle = (1,0)^\top$ and $|1\rangle = (0,1)^\top$.

Exercise 2 *Entanglement by unitary operations*

Let $|x\rangle \otimes |y\rangle$ with $x, y = 0, 1$ be the four states of the "computational basis" (or canonical) for two qubits. We define the unitary operation CNOT (called "controlled-not")

$$\text{CNOT}\,|x\rangle \otimes |y\rangle = |x\rangle \otimes |x \oplus y\rangle \tag{1}$$

where $x \oplus y$ is the addition of bits modulo 2. This operation models certain magnetic interactions between the degree of freedom in spin and is responsible for the entanglement.

1) Verify that

$$|B_{xy}\rangle = (\text{CNOT})(H \otimes I)\,|x\rangle \otimes |y\rangle$$

using Dirac's notation. Is the part $(H \otimes I)\,|x\rangle \otimes |y\rangle$ already entangled? Justify your answer! (Note that the identity above is the reason for the index notation $|B_{xy}\rangle$).

2) Write down CNOT and $H \otimes I$ and their product in matrix notation. Check that the matrices are unitary.

3) From the unitarity of the matrices, give a new proof (compact) of orthonormality of the states $|B_{xy}\rangle$.