Solution 1.

(a) We have:

\[ \Pr(H = 0) \cdot f_{Y_1, Y_2 \mid H}(y_1, y_2) = \begin{cases} \frac{1}{2} \cdot \frac{1}{16\pi} & \text{if } (y_1, y_2) \in B([-1, -1], 4), \\ 0 & \text{otherwise}, \end{cases} \]

and

\[ \Pr(H = 1) \cdot f_{Y_1, Y_2 \mid H}(y_1, y_2) = \begin{cases} \frac{1}{2} \cdot \frac{1}{16\pi} & \text{if } (y_1, y_2) \in B([1, 1], 4), \\ 0 & \text{otherwise}. \end{cases} \]

Therefore, the MAP decision rule requires that:

- We decide \( \hat{H} = 0 \) if \((y_1, y_2) \in B([-1, -1], 4) \setminus B([1, 1], 4)\).
- We decide \( \hat{H} = 1 \) if \((y_1, y_2) \in B([1, 1], 4) \setminus B([-1, -1], 4)\).
- We can decide either \( \hat{H} = 0 \) or \( \hat{H} = 1 \) if \((y_1, y_2) \notin B([-1, -1], 4) \cup B([1, 1], 4)\).

We conclude that the answers for (a) are:

1. T.
2. T.
3. T.

(b) Let \( p = \Pr(H = 0) \). We have:

\[ \Pr(H = 0) \cdot f_{Y_1, Y_2 \mid H}(y_1, y_2) = \begin{cases} p \cdot \frac{1}{16\pi} & \text{if } (y_1, y_2) \in B([-1, -1], 4), \\ 0 & \text{otherwise}, \end{cases} \]

and

\[ \Pr(H = 1) \cdot f_{Y_1, Y_2 \mid H}(y_1, y_2) = \begin{cases} (1 - p) \cdot \frac{1}{16\pi} & \text{if } (y_1, y_2) \in B([1, 1], 4), \\ 0 & \text{otherwise}. \end{cases} \]

Since \( p > 1 - p \), the MAP decision rule requires that:

- We decide \( \hat{H} = 0 \) if \((y_1, y_2) \in B([-1, -1], 4)\).
- We decide \( \hat{H} = 1 \) if \((y_1, y_2) \in B([1, 1], 4) \setminus B([-1, -1], 4)\).
- We can decide either \( \hat{H} = 0 \) or \( \hat{H} = 1 \) if \((y_1, y_2) \notin B([-1, -1], 4) \cup B([1, 1], 4)\).

We conclude that the answers for (b) are:

1. F.
2. F.
(3) T.

(c) Notice that for the case of (b), the two points \([0.5, 0.5]\) and \([-2, 3]\) satisfy \(y_1 + y_2 = 1\). However, the MAP decision rule requires that we decide \(\hat{H} = 0\) if we observe \([0.5, 0.5]\), and it requires that we decide \(\hat{H} = 1\) if we observe \([-2, 3]\). Therefore, no MAP decision rule can be based on the observation of \(Y_1 + Y_2\) alone, hence it is not a sufficient statistic. We conclude that the answer for (c) is F.

Solution 2.

(a)

\[
Z(p) = \sum_y \sqrt{\Pr(Y = y|H = 0)\Pr(Y = y|H = 1)}
= \sqrt{(1 - p)p + p(1 - p)} = \sqrt{4p(1 - p)}.
\]

(b) (1)

\[
Z_n(p) = \sum_{y_1, \ldots, y_n} \sqrt{\prod_{i=1}^n \Pr(Y_i = y_i|H = 0) \Pr(Y_i = y_i|H = 1)}
= \prod_{i=1}^n \left( \sum_{y_i} \sqrt{\Pr(Y_i = y_i|H = 0)\Pr(Y_i = y_i|H = 1)} \right)
= \prod_{i=1}^n \left( \sum_{y_i} \sqrt{\Pr(Y_i = y_i|H = 0)\Pr(Y_i = y_i|H = 1)} \right) = Z(p)^n = (4p(1 - p))^{\frac{n}{2}}.
\]

(2) Let \((y_1, \ldots, y_n)\) be the observation. The MAP decision rule is:

\[
\Pr(H = 0|\Pr(Y_1 = y_1, \ldots, Y_n = y_n|H = 0) \overset{\hat{H}=0}{\geq} \Pr(H = 1|\Pr(Y_1 = y_1, \ldots, Y_n = y_n|H = 1)
\iff \frac{1}{2}p\sum_{i=1}^n y_i (1 - p)^{n - \sum_{i=1}^n y_i} \overset{\hat{H}=0}{\geq} \frac{1}{2}(1 - p)\sum_{i=1}^n y_i p^{n - \sum_{i=1}^n y_i}
\iff \frac{p}{1 - p} \overset{\hat{H}=0}{\geq} \left( \frac{p}{1 - p} \right)^n
\iff \sum_{i=1}^n y_i \overset{\hat{H}=0}{\geq} \frac{n}{2}.
\]

Therefore, a decision rule that minimizes the probability of error is:

\[
\hat{H} = \begin{cases} 0 & \text{if } \sum_{i=1}^n Y_i \leq \frac{n}{2} \\ 1 & \text{if } \sum_{i=1}^n Y_i > \frac{n}{2}. \end{cases}
\]

(c) For the decision rule in (b.2), we have:

\[
\Pr(\text{Error}|H = 0) = \Pr \left( \sum_{i=1}^n Y_i > \frac{n}{2} \Big| H = 0 \right) \overset{(s)}{=} \Pr \left( \sum_{i=1}^n X_i > \frac{n}{2} \right),
\]

2
where (*) follows from the fact that given $H = 0$, $(Y_i)_{1 \leq i \leq n}$ is distributed as $(X_i)_{1 \leq i \leq n}$. On the other hand,

$$\Pr(\text{Error} | H = 1) = \Pr \left( \sum_{i=1}^{n} Y_i \leq \frac{n}{2} | H = 1 \right) = \Pr \left( \sum_{i=1}^{n} (1 - Y_i) \geq \frac{n}{2} | H = 1 \right)$$

where (**) follows from the fact that given $H = 1$, $(1 - Y_i)_{1 \leq i \leq n}$ is distributed as $(X_i)_{1 \leq i \leq n}$. Therefore,

$$\Pr \left( \sum_{i=1}^{n} X_i > \frac{n}{2} \right) \leq \Pr(\text{Error}) \leq Z_n(p) = (4p(1-p))^{\frac{n}{2}}.$$ 

**Solution 3.**

(c) For (a), assume first that $c_0$ was transmitted. The probability of a correct guess is:

$$p_{c,0} = \Pr(\text{Error}) = \Pr(\text{Error} | H = 1) = \Pr \left( \sum_{i=1}^{n} X_i > \frac{n}{2} \right) \leq Z_n(p) = (4p(1-p))^{\frac{n}{2}}.$$
Therefore, the probability of error, assuming $c_0$ was transmitted, is:

$$p_{e,0} = 1 - (1 - Q(\sqrt{2}))^2.$$ 

Now because of the symmetry between the regions, we have $p_{e,1} = p_{e,2} = p_{e,3} = p_{e,0} = 1 - (1 - Q(\sqrt{2}))^2$. Therefore, the probability of error is $p_e = 1 - (1 - Q(\sqrt{2}))^2$.

For (b), assume first that $c_0$ was transmitted. The probability of a correct guess is:

$$p'_{e,0} = \Pr(Z_1 \geq -1, Z_2 \geq -1) = \Pr(Z_1 \geq -1)^2 = (1 - \Pr(Z_1 < -1))^2$$

$$= \left(1 - \int_{-\infty}^{-1} \frac{1}{2} e^{-|z|/2} dz\right)^2 = \left(1 - \frac{1}{2} e^{-1}\right)^2.$$ 

Therefore, the probability of error, assuming $c_0$ was transmitted, is:

$$p'_{e,0} = 1 - \left(1 - \frac{1}{2} e^{-1}\right)^2.$$ 

Because of the symmetry between the regions, we have $p'_{e,1} = p'_{e,2} = p'_{e,3} = p'_{e,0} = 1 - (1 - \frac{1}{2} e^{-1})^2$. Therefore, the probability of error is $p'_e = 1 - (1 - \frac{1}{2} e^{-1})^2$.

We know that for every $x \geq 0$, we have $Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}}$. Thus, $Q(\sqrt{2}) \leq \frac{1}{2} e^{-1}$, hence $p_e \leq p'_e$. We conclude that the probability of error in (a) is less than or equal to that in (b).