ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 16 Principles of Digital Communications

Solutions to Quiz 1 Apr. 06, 2017

SOLUTION 1.

(a) We have:

$$\Pr(H=0)\cdot f_{Y_1,Y_2|H}(y_1,y_2) = \begin{cases} \frac{1}{2}\cdot\frac{1}{16\pi} & \text{if } (y_1,y_2)\in B([-1,-1],4), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\Pr(H=1)\cdot f_{Y_1,Y_2|H}(y_1,y_2) = \begin{cases} \frac{1}{2}\cdot\frac{1}{16\pi} & \text{if } (y_1,y_2)\in B([1,1],4), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the MAP decision rule requires that:

- We decide $\hat{H}=0$ if $(y_1,y_2)\in B([-1,-1],4)\setminus B([1,1],4)$.
- We decide $\hat{H}=1$ if $(y_1,y_2)\in B([1,1],4)\setminus B([-1,-1],4)$.
- We can decide either $\hat{H}=0$ or $\hat{H}=1$ if $(y_1,y_2)\in B([-1,-1],4)\cap B([1,1],4)$ or $(y_1,y_2)\notin B([-1,-1],4)\cup B([1,1],4)$.

We conclude that the answers for (a) are:

(1) **T**.

(2) **T**.

(3) **T**.

(b) Let $p=\Pr(H=0)$. We have:

$$\Pr(H=0)\cdot f_{Y_1,Y_2|H}(y_1,y_2) = \begin{cases} p\cdot\frac{1}{16\pi} & \text{if } (y_1,y_2)\in B([-1,-1],4), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\Pr(H=1)\cdot f_{Y_1,Y_2|H}(y_1,y_2) = \begin{cases} (1-p)\cdot\frac{1}{16\pi} & \text{if } (y_1,y_2)\in B([1,1],4), \\ 0 & \text{otherwise.} \end{cases}$$

Since $p>1-p$, the MAP decision rule requires that:

- We decide $\hat{H}=0$ if $(y_1,y_2)\in B([-1,-1],4)$.
- We decide $\hat{H}=1$ if $(y_1,y_2)\in B([1,1],4)\setminus B([-1,-1],4)$.
- We can decide either $\hat{H}=0$ or $\hat{H}=1$ if $(y_1,y_2)\notin B([-1,-1],4)\cup B([1,1],4)$.

We conclude that the answers for (b) are:

(1) **F**.

(2) **F**.

(3) **T**.

(c) Notice that for the case of (b), the two points $[0.5, 0.5]$ and $[-2, 3]$ satisfy $y_1 + y_2 = 1$. However, the MAP decision rule requires that we decide $\hat{H} = 0$ if we observe $[0.5, 0.5]$, and it requires that we decide $\hat{H} = 1$ if we observe $[-2, 3]$. Therefore, no MAP decision rule can be based on the observation of $Y_1 + Y_2$ alone, hence it is not a sufficient statistic. We conclude that the answer for (c) is **F**.

SOLUTION 2.

(a)

$$\begin{aligned} Z(p) &= \sum_y \sqrt{\Pr(Y = y|H = 0)\Pr(Y = y|H = 1)} \\ &= \sqrt{(1-p)p} + \sqrt{p(1-p)} = \sqrt{4p(1-p)}. \end{aligned}$$

(b) (1)

$$\begin{aligned} Z_n(p) &= \sum_{y_1,\ldots,y_n} \sqrt{\Pr(Y_1 = y_1, \ldots, Y_n = y_n|H = 0)\Pr(Y_1 = y_1, \ldots, Y_n = y_n|H = 1)} \\ &= \sum_{y_1,\ldots,y_n} \sqrt{\left(\prod_{i=1}^n \Pr(Y_i = y_i|H = 0)\right)\left(\prod_{i=1}^n \Pr(Y_i = y_i|H = 1)\right)} \\ &= \sum_{y_1,\ldots,y_n} \prod_{i=1}^n \sqrt{\Pr(Y_i = y_i|H = 0)\Pr(Y_i = y_i|H = 1)} \\ &= \prod_{i=1}^n \left(\sum_{y_i} \sqrt{\Pr(Y_i = y_i|H = 0)\Pr(Y_i = y_i|H = 1)}\right) = Z(p)^n = (4p(1-p))^{\frac{n}{2}}. \end{aligned}$$

(2) Let $(y_1, \ldots, y_n)$ be the observation. The MAP decision rule is:

$$\begin{aligned} \Pr(H = 0)\Pr(Y_1 = y_1, \ldots, Y_n = y_n|H = 0) &\underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} \Pr(H = 1)\Pr(Y_1 = y_1, \ldots, Y_n = y_n|H = 1) \\ \Leftrightarrow \quad \frac{1}{2}p^{\sum_{i=1}^n y_i}(1-p)^{n-\sum_{i=1}^n y_i} &\underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} \frac{1}{2}(1-p)^{\sum_{i=1}^n y_i}p^{n-\sum_{i=1}^n y_i} \\ \Leftrightarrow \quad \left(\frac{p}{1-p}\right)^{2\sum_{i=1}^n y_i} &\underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} \left(\frac{p}{1-p}\right)^n \\ \Leftrightarrow \quad \sum_{i=1}^n y_i &\underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{n}{2}. \end{aligned}$$

Therefore, a decision rule that minimizes the probability of error is:

$$\hat{H} = \begin{cases} 0 & \text{if } \sum_{i=1}^n Y_i \le \frac{n}{2} \\ 1 & \text{if } \sum_{i=1}^n Y_i > \frac{n}{2}. \end{cases}$$

(c) For the decision rule in (b.2), we have:

$$\Pr(\text{Error}|H = 0) = \Pr\left(\sum_{i=1}^n Y_i > \frac{n}{2}\middle| H = 0\right) \overset{(*)}{=} \Pr\left(\sum_{i=1}^n X_i > \frac{n}{2}\right),$$

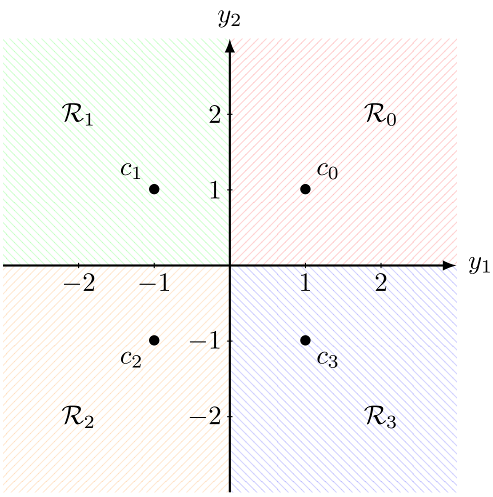
where $(*)$ follows from the fact that given $H=0$, $(Y_i)_{1\leq i\leq n}$ is distributed as $(X_i)_{1\leq i\leq n}$. On the other hand,

$$\Pr(\text{Error}|H=1) = \Pr\left(\sum_{i=1}^{n} Y_i \leq \frac{n}{2}\middle| H=1\right) = \Pr\left(\sum_{i=1}^{n}(1-Y_i) \geq \frac{n}{2}\middle| H=1\right)$$
$$\overset{(**)}{=} \Pr\left(\sum_{i=1}^{n} X_i \geq \frac{n}{2}\right) \geq \Pr\left(\sum_{i=1}^{n} X_i > \frac{n}{2}\right),$$

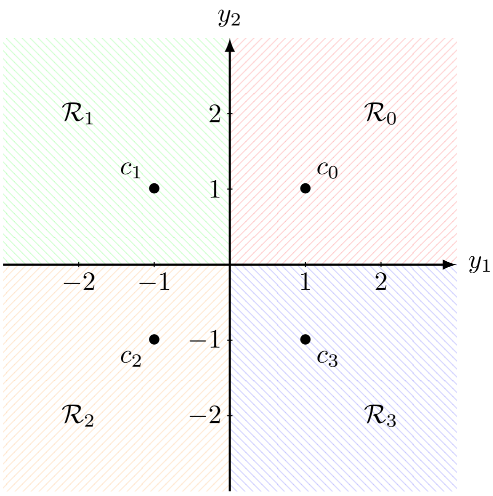
where $(**)$ follows from the fact that given $H=1$, $(1-Y_i)_{1\leq i\leq n}$ is distributed as $(X_i)_{1\leq i\leq n}$. Therefore,

$$\Pr\left(\sum_{i=1}^{n} X_i > \frac{n}{2}\right) \leq \Pr(\text{Error}) \leq Z_n(p) = (4p(1-p))^{\frac{n}{2}}.$$

Solution 3.



(a)



(b)

(c) For (a), assume first that c_0 was transmitted. The probability of a correct guess is:

$$p_{c,0} = \Pr(Z_1 \geq -1, Z_2 \geq -1) = Q\left(-\frac{1}{\sqrt{\frac{1}{2}}}\right)^2 = (1-Q(\sqrt{2}))^2.$$

Therefore, the probability of error, assuming c_0 was transmitted, is:

$$p_{e,0} = 1 - (1 - Q(\sqrt{2}))^2.$$

Now because of the symmetry between the regions, we have $p_{e,1} = p_{e,2} = p_{e,3} = p_{e,0} = 1 - (1 - Q(\sqrt{2}))^2$. Therefore, the probability of error is $p_e = 1 - (1 - Q(\sqrt{2}))^2$.

For (b), assume first that c_0 was transmitted. The probability of a correct guess is:

$$\begin{aligned} p'_{c,0} &= \Pr(Z_1 \geq -1, Z_2 \geq -1) = \Pr(Z_1 \geq -1)^2 = (1 - \Pr(Z_1 < -1))^2 \\ &= \left(1 - \int_{-\infty}^{-1} \frac{1}{2} e^{-|z|} dz\right)^2 = \left(1 - \frac{1}{2} e^{-1}\right)^2. \end{aligned}$$

Therefore, the probability of error, assuming c_0 was transmitted, is:

$$p'_{e,0} = 1 - \left(1 - \frac{1}{2} e^{-1}\right)^2.$$

Because of the symmetry between the regions, we have $p'_{e,1} = p'_{e,2} = p'_{e,3} = p'_{e,0} = 1 - (1 - \frac{1}{2} e^{-1})^2$. Therefore, the probability of error is $p'_e = 1 - (1 - \frac{1}{2} e^{-1})^2$.

We know that for every $x \geq 0$, we have $Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}}$. Thus, $Q(\sqrt{2}) \leq \frac{1}{2} e^{-1}$, hence $p_e \leq p'_e$. We conclude that the probability of error in (a) is less than or equal to that in (b).