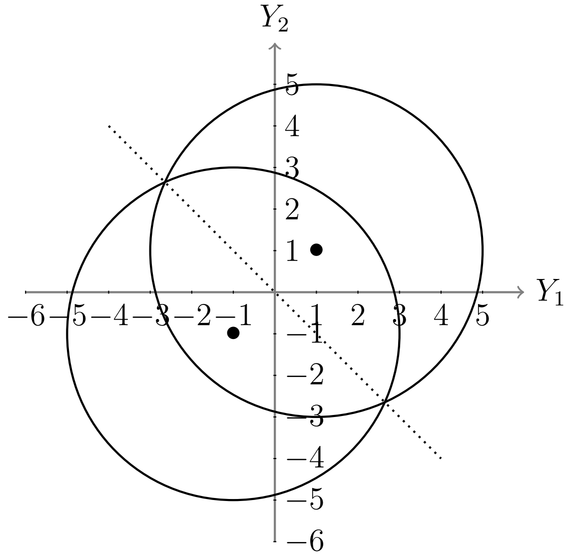
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 10 Principles of Digital Communications
Quiz 1 Mar. 22, 2017

PROBLEM 1. Let $B(\mathbf{x}, r)$ denote the disc of radius r and center $\mathbf{x}$ in $\mathbb{R}^2$. Assume we only have two hypotheses, $H = 0$ or $H = 1$. If $H = 0$, the observation $\mathbf{Y} = (Y_1, Y_2)$ is uniformly distributed on $B([-1, -1], 4)$. Otherwise, if $H = 1$, $\mathbf{Y}$ is uniformly distributed on $B([1, 1], 4)$. The following questions are True($\mathbf{T}$)/False($\mathbf{F}$) questions. No justifications needed. Grading: +1 for correct answer, 0 point for no answer, -1 for incorrect answer.



(a) Assuming the two hypotheses are equally probable, does each one of these tests minimize the probability of error:

(1) $Y_1 + Y_2 \underset{0}{\overset{1}{\gtrless}} 0$ **T F**

(2) $\hat{H} = \begin{cases} 1 & \text{if } \mathbf{Y} \in B([1,1],4) \\ 0 & \text{otherwise} \end{cases}$ **T F**

(3) $\hat{H} = \begin{cases} 1 & \text{if } \mathbf{Y} \notin B([-1,-1],4) \\ 0 & \text{otherwise} \end{cases}$ **T F**

(b) Now, suppose H_0 is more probable than H_1, i.e. $\Pr(H = 0) > \Pr(H = 1)$. Re-do a).

(1) $Y_1 + Y_2 \underset{0}{\overset{1}{\gtrless}} 0$ **T F**

(2) $\hat{H} = \begin{cases} 1 & \text{if } \mathbf{Y} \in B([1,1],4) \\ 0 & \text{otherwise} \end{cases}$ **T F**

(3) $\hat{H} = \begin{cases} 1 & \text{if } \mathbf{Y} \notin B([-1,-1],4) \\ 0 & \text{otherwise} \end{cases}$ **T F**

(c) $T = Y_1 + Y_2$ is a sufficient statistic. **T F**

PROBLEM 2. Suppose the observation Y is binary. Under Hypothesis 0, $\Pr(Y = 1) = p$, and under Hypothesis 1, $\Pr(Y = 0) = p$, $(p < 1/2)$. The two hypotheses are equally likely.

(a) Find the Bhattacharrya parameter $Z(p)$ in this setting.

(b) Suppose now we observe $(Y_1, \ldots, Y_n)$ where $\{Y_i\}$ are independent, each distributed like Y above, $n \geq 1$.

(1) Find the Bhattacharrya parameter $Z_n(p)$ in this setting.

(2) Find the decision rule that minimizes the probability of error.

(c) Show that if $X_1, X_2, \ldots$ are i.i.d. with $X_i = \begin{cases} 0 & \text{with probability } 1-p \\ 1 & \text{with probability } p \end{cases}$,

then $\Pr(\sum_{i=1}^{n} X_i > n/2) \leq (4p(1-p))^{n/2}$.

PROBLEM 3. Four equally likely messages, ($i = 0, 1, 2, 3$), are mapped to the points on the plane $(\pm 1, \pm 1)$. The transmitted point is corrupted by additive noise $Z = (Z_1, Z_2)$ and received as $Y = c_i + Z$.

(a) If Z_i are independent $\mathcal{N}(0, \frac{1}{2})$, sketch the decision regions in Fig. a.

(b) Suppose now Z_i are independent Laplacians, i.e.,

$$f_Z(z_1, z_2) = \frac{1}{2} e^{-|z_1|} \frac{1}{2} e^{-|z_2|} = \frac{1}{4} e^{-(|z_1|+|z_1|)}.$$

Sketch the decision regions in Fig. b.

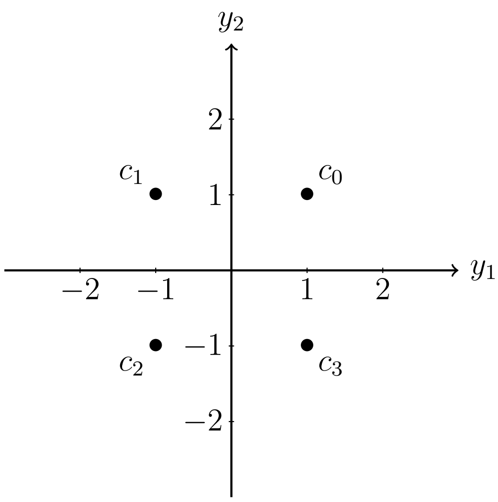


Fig. a: Decision regions for question 3-a).

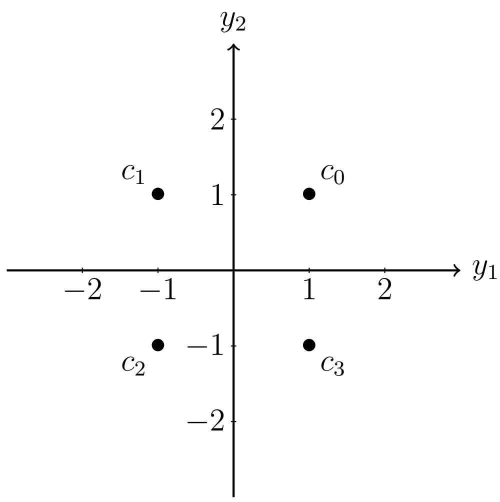


Fig. b: Decision regions for question 3-b).

(c) Compare the error probabilities in (a) and (b).