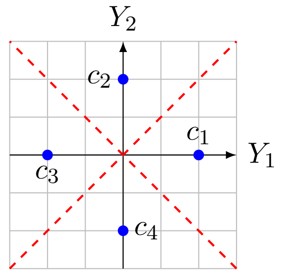
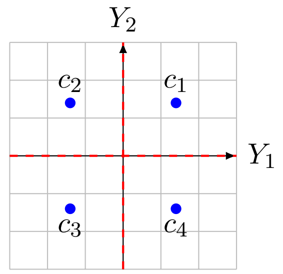
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

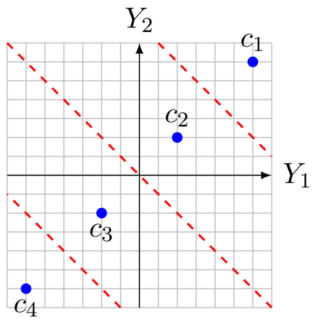
SOLUTION 1.

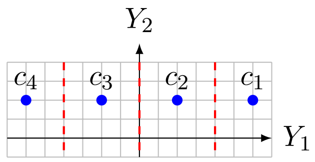
(i) $\boxed{a = \sqrt{3}}$ since,

$$1 = \|\psi\| = \int_0^{\frac{1}{2}} (2at)^2 \mathrm{d}t + \int_{\frac{1}{2}}^{1} (2a(1-t))^2 \mathrm{d}t = \left[\frac{4a^2t^3}{3}\right]_{t=0}^{t=\frac{1}{2}} + \left[\frac{4a^2(t-1)^3}{3}\right]_{t=\frac{1}{2}}^{t=1} = \frac{a^2}{3} = 1$$

<table>
<tr><th></th><th>(ii) Constellation and Decision Regions</th><th>(iii) Probability of Error</th></tr>
<tr><td>(a)</td><td>

$c_1 = (1, 0)$, $c_2 = (0, 1)$, $c_3 = (-1, 0)$, $c_4 = (0, -1)$</td><td>$\boxed{P_e = 2Q\left(1/\sqrt{N_0}\right) - Q\left(1/\sqrt{N_0}\right)^2}$
(4-QPSK constellation rotated by 45°)</td></tr>
<tr><td>(b)</td><td>

$c_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $c_2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $c_3 = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $c_4 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$</td><td>$\boxed{P_e = 2Q\left(1/\sqrt{N_0}\right) - Q\left(1/\sqrt{N_0}\right)^2}$
(same constellation as (a), rotated by 45°)</td></tr>
<tr><td>(c)</td><td>

$c_1 = (3, 3)$, $c_2 = (1, 1)$, $c_3 = (-1, -1)$, $c_4 = (-3, -3)$</td><td>$\boxed{P_e = \frac{3}{2}Q\left(\frac{2}{\sqrt{N_0}}\right)}$
(4-PAM constellation)</td></tr>
<tr><td>(d)</td><td>

$c_1 = (3, 1)$, $c_2 = (1, 1)$, $c_3 = (-1, 1)$, $c_4 = (-3, 1)$</td><td>$\boxed{P_e = \frac{3}{2}Q\left(\frac{\sqrt{2}}{\sqrt{N_0}}\right)}$
(same as (c) contracted by a factor of $\frac{1}{\sqrt{2}}$)</td></tr>
</table>

Note: In each case, c_i denotes the codeword corresponding to w_i. The observation will be $Y = c_i + Z$ where $Z \sim \mathcal{N}(0, \frac{N_0}{2} I_2)$ is white Gaussian noise, thus, the optimal decision is minimum distance decoding.

SOLUTION 2.

(i) Since $\xi(t) = \text{tri}(t-1) - \text{tri}(t-3)$, we have

$$\begin{aligned}\xi_{\mathcal{F}}(f) &= \mathcal{F}\{\text{tri}(t-1) - \text{tri}(t-3)\} = e^{-\mathrm{j}2\pi f}\,\text{sinc}^2(f) - e^{-\mathrm{j}6\pi f}\,\text{sinc}^2(f) \\ &= e^{-\mathrm{j}4\pi f}\,\text{sinc}^2(f)[e^{\mathrm{j}2\pi f} - e^{-\mathrm{j}2\pi f}] = \boxed{2e^{-\mathrm{j}4\pi f}\,\mathrm{j}\sin(2\pi f)\,\text{sinc}^2(f)}.\end{aligned}$$

(ii) We firstly have that $\mathbb{E}[X_i] = \mathbb{E}[D_{2i-1}] + \alpha\,\mathbb{E}[D_{2i}] = 0$ and, since D_{2i-1} and D_{2i} are independent, $\mathbb{E}[|X_i|^2] = \mathbb{E}[D_{2i-1}^2] + \alpha^2\,\mathbb{E}[D_{2i}^2] = 1 + \alpha^2$. Moreover X_i and X_j are independent (for $i \neq j$), since (D_{2i-1}, D_{2i}) is independent of (D_{2j-1}, D_{2j}). Thus,

$$K_X[k] = \mathbb{E}[X_{i+k}X_i^*] = \begin{cases}\mathbb{E}[|X_i|^2] & k = 0; \\ 0 & \text{otherwise.}\end{cases} = (1+\alpha^2)\mathbb{1}\{k=0\}.$$

Therefore,

$$\begin{aligned}S_X(f) &= |\xi_{\mathcal{F}}(f)|^2 \sum_k K_X[k]\exp(-\mathrm{j}2\pi k f) \\ &= (1+\alpha^2)|\xi_{\mathcal{F}}(f)|^2 = \boxed{4(1+\alpha^2)\sin^2(2\pi f)\,\text{sinc}^4(f)}.\end{aligned}$$

The above vanishes at frequencies $f = \frac{m}{2}$, $m \in \mathbb{Z}$.

(iii) In this case we have

$$\begin{aligned}K_X[k] &= \mathbb{E}[X_{i+k}X_i^*] \\ &= \mathbb{E}[D_{i-2+k}D_{i-2}] + \alpha\,\mathbb{E}[D_{i-2+k}D_i] + \alpha\,\mathbb{E}[D_{i+k}D_{i-2}] + \alpha^2\,\mathbb{E}[D_{i+k}D_i] \\ &= (1+\alpha^2)\mathbb{1}\{k=0\} + \alpha\mathbb{1}\{k=2\} + \alpha\mathbb{1}\{k=-2\}.\end{aligned}$$

Therefore,

$$\sum_k K_X[k]\exp(-\mathrm{j}2\pi k f) = (1+\alpha^2) + \alpha[e^{\mathrm{j}4\pi f} + e^{-\mathrm{j}4\pi f}] = (1+\alpha^2) + 2\alpha\cos(4\pi f),$$

and

$$\begin{aligned}S_X(f) &= |\xi_{\mathcal{F}}(f)|^2 \sum_k K_X[k]\exp(-\mathrm{j}2\pi k f) \\ &= \boxed{4[(1+\alpha^2) + 2\alpha\cos(4\pi f)]\sin^2(2\pi f)\,\text{sinc}^4(f)}.\end{aligned}$$

In addition to integer multiples of $\frac{1}{2}$, the above can be equal to zero at f such that

$$\cos(4\pi f) = -\frac{1+\alpha^2}{2\alpha}.$$

(iv) From the solution of (ii), it is obvious that $S_X(\frac{1}{4}) = (1+\alpha^2)\,\text{sinc}^4(\frac{1}{4}) > 0$ regardless of the value of $\alpha \in \mathbb{R}$.

However, in (iii) if we choose $\alpha = 1$, we will have zeros at $f = \frac{2m+1}{4}$, $m \in \mathbb{Z}$ (including $f = \frac{1}{4}$) since $\cos(4\pi f) = \cos((2m+1)\pi) = -1 = -(1+\alpha^2)/(2\alpha)$.

SOLUTION 3. This is in fact the same convolutional code we considered in class, except for the fact that the upper and lower branches are exchanged. Therefore the encoding circuit (except for the switching of the even and odd bits), the detour flow diagram, and the generating function counting detours are identical.

(i) The first 10 bits of the output are $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 1, x_5 = 1, x_6 = 0, x_7 = 1, x_8 = 0, x_9 = 1, x_{10} = 0, \dots$.

(ii) We have $T(I, D) = \frac{ID^5}{1-2ID}$. To check, we see that the first few terms are $ID^5+2I^2D^6+4I^3D^7+\cdots$. The very first term corresponds to the input of one 1 followed by two 0s.

(iii) As stated, we can assume that d is even, so that $d/2$ is an integer. In more detail the derivation of the bound is the following.

$$
\begin{aligned}
P\{u \to v\} &\overset{(a)}{\leq} \sum_{e=d/2}^{d} \binom{d}{e} \epsilon^e (1-\epsilon)^{d-e} \\
&\overset{(b)}{=} (1-\epsilon)^d \sum_{e=d/2}^{d} \binom{d}{e} \left(\epsilon/(1-\epsilon)\right)^e \\
&\overset{(c)}{\leq} (1-\epsilon)^d (\epsilon/(1-\epsilon))^{d/2} \sum_{e=d/2}^{d} \binom{d}{e} \\
&\overset{(d)}{\leq} (1-\epsilon)^d (\epsilon/(1-\epsilon))^{d/2} 2^d/2 \\
&\overset{(e)}{=} \frac{1}{2} \underbrace{\left(2\sqrt{(1-\epsilon)\epsilon}\right)}_{\mathcal{B}}^{d}.
\end{aligned}
$$

Step (a) follows since we will make a mistake only if we flip at least $d/2$ of the bits where u and v differ. For step (b) we just took out the common term $(1-\epsilon)^d$. Step (c) follows since every term in the sum contains at least the factor $(\epsilon/(1-\epsilon))^{d/2}$ and $\epsilon/(1-\epsilon) < 1$. Step (d) follows since $\sum_{e=0}^{d} \binom{d}{e} = 2^d$, the terms of this sum are symmetric around $e = d/2$, and we include slightly more than half this total sum. For step (e) we just collected terms to clean up the result.

(iv) The upper bound is identical to the one we derived during class for the AWGNC, except that we have to use now for z the proper Bhattacharyya value, namely $z = 2\sqrt{\epsilon(1-\epsilon)}$ (instead of $z = e^{-\frac{\mathcal{E}_s}{2\sigma^2}}$), and that we can add a factor $1/2$ if we want a tighter bound. In total we get

$$
P_b \leq \frac{1}{2} \frac{\partial T(I,D)}{\partial I}\Big|_{I=1;D=z}.
$$

Solution 4.

(i) Using the Parseval identity, $\langle w_{E,0}, w_{E,1}\rangle = \int_t w_{E,0}(t)w^*_{E,1}(t)\mathrm{d}t = \int w_{E,0,\mathcal{F}}(f)w^*_{E,1,\mathcal{F}}(f)\mathrm{d}f$, $w_{E,0,\mathcal{F}}(f) = \mathrm{rect}(f)$, and $w_{E,1,\mathcal{F}}(f) = \mathrm{rect}(f-f_0)$. Thus if $f_0 \geq 1$, $\mathrm{rect}(f)$ and $\mathrm{rect}(f-f_0)$ do not overlap and $w_{E,0}$ and $w_{E,1}$ will be orthogonal.

(ii) We know that

$$\text{under } H = i: \quad R_E(t) = w_{E,i}(t) + N_E(t)$$

(where $N_E(t)$ is complex-valued white Gaussian noise). For the optimal decision one has to project the received signal onto the orthonormal basis spanned by $\{w_{E,0}, w_{E,1}\}$. In this case since $w_{E,0}$ and $w_{E,1}$ are orthogonal themselves (and both have unit norm) the basis waveforms are simply $\{w_{E,0}, w_{E,1}\}$. For Y_1 to be the projection of the received signal on $w_{E,0}$ we need to have

$$h_1(t) = w^*_{E,0}(1-t) = \mathrm{sinc}(1-t).$$

For the similar reason,

$$h_2(t) = w^*_{E,1}(1-t) = \exp(\mathtt{j}2\pi f_0(1-t))\,\mathrm{sinc}(1-t) = \exp(-\mathtt{j}2\pi f_0 t)\,\mathrm{sinc}(1-t),$$

(since $f_0 = 1$).

(iii) Using the matched filters we found in (ii) the hypothesis testing problem can be formulated as

$$\text{under } H = i: \quad (Y_1, Y_2) = c_i + (Z_1, Z_2),$$

where $Z = (Z_1, Z_2) \sim \mathcal{N}_{\mathcal{C}}(0, N_0 I_2)$ and $c_1 = (1,0)$ and $c_2 = (0,1)$. Since the imaginary parts of c_1 and c_2 are identical, the distribution of $\Im\{Y\} = (\Im\{Y_1\}, \Im\{Y_2\})$ remains the same under either hypothesis, thus, it is irrelevant. The MAP decision rule is minimum distance (since the two hypotheses are equally likely):

$$\hat{H}_{\mathrm{MAP}}(Y_1, Y_2) = \begin{cases} 0 & \text{if } \Re\{Y_1\} \geq \Re\{Y_2\}; \\ 1 & \text{if } \Re\{Y_1\} \leq \Re\{Y_2\}. \end{cases}$$

(iv) The noiseless output of the Hilbert filter is $\hat{w}_i(t) = w_{E,i}(t)\exp(\mathtt{j}2\pi f_c t)$ (since $w_i(t) = \frac{\sqrt{2}}{2}\left[w_{E,i}(t)\exp(\mathtt{j}2\pi f_c t) + w^*_{E,i}(t)\exp(-\mathtt{j}2\pi f_c t)\right]$ and the filter $h_>(t)$ removes the second term because it lies on the negative part of the frequency spectrum). Thus, in presence of a phase difference, the noiseless output of the demodulator equals

$$\exp(-\mathtt{j}(2\pi f_c t + \theta))\hat{w}_i(t) = \exp(-\mathtt{j}\theta)w_{E,i}(t)$$

Therefore,

$$\text{under } H = i: \quad R_E(t) = w_{E,i}(t)\exp(-\mathtt{j}\theta) + N_E(t),$$

(where $N_E(t)$ is complex-valued white Gaussian noise). Hence,

$$\text{under } H = i: \quad (Y_1, Y_2) = \exp(-\mathtt{j}\theta)c_i + (Z_1, Z_2), \qquad i = 0, 1,$$

and (since c_i's are real-valued),

$$\text{under } H = i: \quad (\Re\{Y_1\}, \Re\{Y_2\}) = \cos(\theta)c_i + (Z_{1,R}, Z_{2,R}), \qquad i = 0, 1,$$

where $(Z_{1,R}, Z_{2,R}) \sim \mathcal{N}(0, \frac{N_0}{2} I_2)$. Consequently,

$$\begin{aligned}\Pr\{\text{error}|H=0\} &= \Pr\{\cos(\theta) + Z_{1,R} \le Z_{2,R}\} \\ &= \Pr\{Z_{2,R} - Z_{1,R} \ge \cos(\theta)\} \\ &= Q(\cos(\theta)/\sqrt{N_0}),\end{aligned}$$

where the last equality follows since $Z_{2,R} - Z_{1,R} \sim \mathcal{N}(0, N_0)$. Similarly we get

$$\Pr\{\text{error}|H=1\} = Q(\cos(\theta)/\sqrt{N_0})$$

thus, $P_e = Q(\cos(\theta)/\sqrt{N_0})$. We see that as $\theta \to \frac{\pi}{2}$, $P_e \to Q(0) = \frac{1}{2}$ (this means that the receiver is not performing better than a random coin-flip!).

(v) It is obvious that when we have perfect synchronization (i.e., $\theta = 0$), the receiver using $|\cdot|$ instead of $\Re\{\cdot\}$ cannot have a better performance than what we found in (iii) because the latter is the MAP rule which is *optimal*.

(vi) Let us look at $|Y_1|^2$ under the hypothesis $H = i$:

$$\begin{aligned}|Y_1|^2 &= \langle Y_1, Y_1^* \rangle \\ &= \langle e^{-\mathrm{j}\theta} c_{i,1} + Z_1, e^{\mathrm{j}\theta} c_{i,1} + Z_1^* \rangle \\ &= c_{i,1}^2 + |Z_1|^2 + 2c_{i,1} \Re\{e^{\mathrm{j}\theta} Z_1\}\end{aligned}$$

where we have used the fact that $c_{i,1} \in \mathbb{R}$. In the above, the only term that depends on θ is the last term. We argue that the distribution of $2\Re\{e^{\mathrm{j}\theta} c_{i,1} Z_1\}$ is independent of θ. Since $Z_1 = Z_{1,R} + \mathrm{j} Z_{1,I}$,

$$2c_{i,1}\Re\{e^{\mathrm{j}\theta} Z_1\} = 2c_{i,1} \left[\cos(\theta) Z_{1,R} - \sin(\theta) Z_{1,I}\right].$$

As $Z_{1,R}$ and $Z_{1,I}$ are independent $\mathcal{N}(0, \frac{N_0}{2})$ random variables, the term inside the square brackets is a $\mathcal{N}(0, \frac{N_0}{2})$ random variable (whose distribution is obviously independent of θ). A similar reasoning shows that the distribution of $|Y_2|^2$ is also independent of θ. Since, conditioned on $H = i$, Y_1 and Y_2 are statistically independent we conclude that the distribution of $(|Y_1|^2, |Y_2|^2)$ does not depend on θ. Thus this sub-optimal receiver is insensitive to phase offsets.