ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences



- 3 problems, 75 points, 165 minutes
- This is a closed book exam.
- Only one two-sided handwritten A4 page of summary allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET

PROBLEM 1. (20 points) Suppose that the relationship between the input $(x_1, x_2) \in \mathbb{R}^2$ and output $(Y_1, Y_2) \in \mathbb{R}^2$ of a communication channel is given by

$$\begin{aligned} Y_1 &= H_{11}x_1 + H_{12}x_2 + Z_1, \\ Y_2 &= H_{21}x_1 + H_{22}x_2 + Z_2, \end{aligned}$$

where $H_{11}, H_{12}, H_{21}, H_{22}, Z_1, Z_2$ are i.i.d., and $\mathcal{N}(0,1)$.

(a) Show that when $x = (x_1, x_2)$ is the input, the output $Y = (Y_1, Y_2)$ is distributed as $\mathcal{N}(0, (1 + \|x\|^2)I)$, where $\|x\|^2 = x_1^2 + x_2^2$.

(b) Suppose $m$ messages are mapped to $m$ codewords $c_0, \dots, c_{m-1}$ to be inputs to the channel above. Show that $T = Y_1^2 + Y_2^2$ is a sufficient statistic.

(c) Show that when $c_i$ is sent, $T$ is exponentially distributed with rate $\lambda_i = \dfrac{1}{2(1 + \|c_i\|^2)}$. (I.e., $\Pr(T > t | H = i) = e^{-\lambda_i t}$ for every $t \geq 0$.)

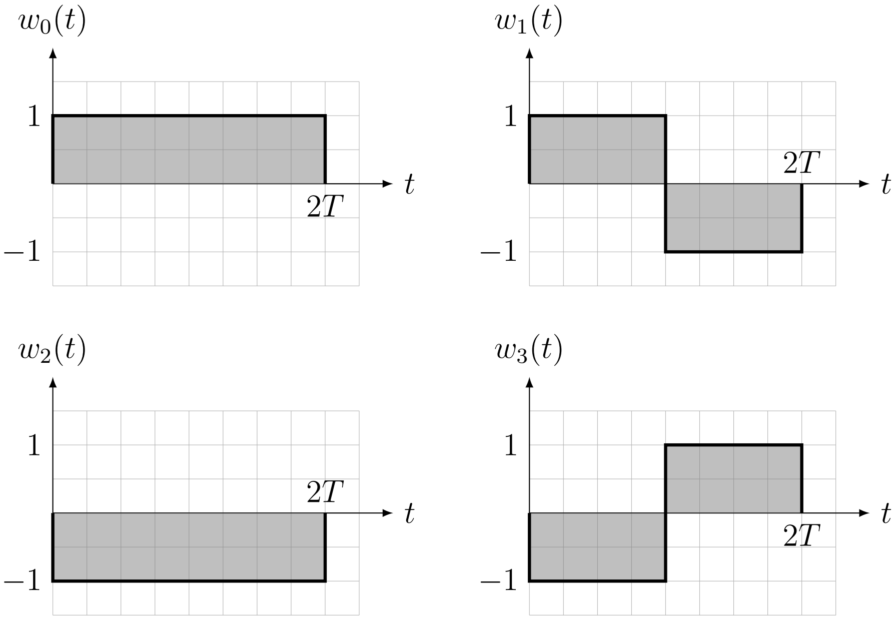
Consider now the following four constellations to transmit 1 bit of information over this channel:

$$\begin{aligned} 1 &: \; c_0 = (0,0), \; c_1 = (5,0). \\ 2 &: \; c_0 = (5,0), \; c_1 = (0,5). \\ 3 &: \; c_0 = (5,0), \; c_1 = (-5,0). \\ 4 &: \; c_0 = (0,0), \; c_1 = (4,3). \end{aligned}$$

Let $P_{e1}, P_{e2}, P_{e3}, P_{e4}$ be the error probabilities of the these four constellations when the bit sent is equally likely to be 0 or 1.

(d) What can you say about the ordering of $P_{e1}, P_{e2}, P_{e3}, P_{e4}, 0, \frac{1}{2}$ and 1. I.e., what equality and inequality relations are there between $P_{ei}$ and $P_{ej}$, and how each probability of error compare to $0, \frac{1}{2}$ and 1 respectively?

PROBLEM 2. (25 points) Consider the following set of waveforms to send two bits of information:




The waveforms are sent over an AWGN channel with noise spectral density $N_0/2$. The four waveforms are equiprobable.

(a) Find an orthonormal basis for the span of $\{w_0, w_1, w_2, w_3\}$ in which the basis functions are time shifts of each other.

(b) Draw an optimal receiver implementation that uses only a single matched filter.

(c) Find the error probability (in terms of $T$ and $N_0$) of the implementation in (b).

(d) Suppose that the receiver's clock is wrong by $T/2$ seconds and the matched filter output is sampled with $T/2$ seconds delay, but the receiver is unaware of this fact. Find the error probability.

(e) Suppose that the receiver's clock is correct, but the transmitter's clock runs at twice the rate, i.e., the transmitted signals are $\tilde{w}_i(t) = w_i(2t)$ instead of $w_i(t)$. Find the error probability.

Problem 3. (30 points) Suppose we have a constellation with $m = 3$ points:

$$c_0 = (0, 0, 0), \quad c_1 = (-2, 2, 0) \text{ and } c_2 = (-2, 0, 2).$$

As usual, transmitted waveforms are given by

$$w_i(t) = \sum_{j=1}^{3} c_{ij} \phi_j(t).$$

However, while $\|\phi_1\| = \|\phi_2\| = \|\phi_3\| = 1$, the signals $\phi_1, \phi_2, \phi_3$ are not orthogonal, and we have $\langle \phi_i, \phi_j \rangle = -1/2$ for $i \neq j$.

(a) Find $\|\phi_1 + \phi_2 + \phi_3\|^2$. Are $\phi_1, \phi_2, \phi_3$ linearly independent?

(b) Apply the Gram-Schmidt procedure on $\{\phi_1, \phi_2, \phi_3\}$ to find a basis $\{\psi_i : i = 1, ..., n\}$. What is the dimension $n$ that results from this procedure?

(c) What is the signal constellation $\tilde{c}_0, \tilde{c}_1, \tilde{c}_2$ in the $\psi$-basis? (I.e., find $\tilde{c}_i = (\tilde{c}_{i1}, \ldots, \tilde{c}_{in})$ so that $w_i(t) = \sum_{j=1}^{n} \tilde{c}_{ij} \psi_j(t)$.)

(d) Find a translation of the constellation so that it has minimum average energy.

(e) Suppose $f_0(y)$, $f_1(y)$, $f(y)$ are the probability density functions of Gaussian random vectors $N(\mu_0, \sigma^2 I), N(\mu_1, \sigma^2 I)$ and $N\left(\frac{\mu_0+\mu_1}{2}, \sigma^2 I\right)$. Show that

$$f_0(y) f_1(y) = f(y)^2 \exp\left(-\frac{\|\mu_0 - \mu_1\|^2}{4\sigma^2}\right).$$

(f) Assuming that the three waveforms are equiprobable, and assuming that the transmission takes place over an AWGN channel with noise spectral density $1/2$, find an upper bound on the probability of error of the constellation in (d) using the union Bhattacharyya bound.