Solution 1.

(a) Given that $H_{11}, H_{12}, H_{21}, H_{22}, Z_1, Z_2$ are i.i.d., then Y_1 and Y_2 are i.i.d. Moreover, given $x = (x_1, x_2)$, $Y_i = H_{i1}x_1 + H_{i2}x_2 + Z_i$ is the sum of three independent Gaussian random variables distributed as $\mathcal{N}(0, x_1^2)$, $\mathcal{N}(0, x_2^2)$ and $\mathcal{N}(0, 1)$. Hence, Y_i is a Gaussian random variable with zero mean and variance equal to $1+x_1^2+x_2^2 = 1+\|x\|^2$. Therefore, $Y = (Y_1, Y_2)$ is distributed as $\mathcal{N}(0, (1 + \|x\|^2)I)$.

(b) We use the Neyman–Fisher factorization theorem to prove that $T = Y_1 + Y_2$ is a sufficient statistic. For that we denote by H the hypothesis on the message. Hence, the likelihood probability given that $H = i$ can be written as

$$f_{Y|H}(y|i) = \underbrace{\frac{1}{2\pi\left(1 + \|c_i\|^2\right)} \exp\left(-\frac{\|y\|^2}{2\left(1 + \|c_i\|^2\right)}\right)}_{g_i\left(Y_1^2+Y_2^2\right)} \times \underbrace{1}_{h(Y)}$$

(c)

$$\begin{aligned}
\mathbb{P}\left(T > t|H = i\right) &\overset{(a)}{=} \int\int_D f_{(Y_1,Y_2)|H}(y_1, y_2|i)\mathrm{d}y_1\mathrm{d}y_2 \\
&= \int\int_D \frac{1}{2\pi(1 + \|c_i\|^2)} \exp\left(-\frac{y_1^2 + y_2^2}{2(1 + \|c_i\|^2)}\right) \mathrm{d}y_1\mathrm{d}y_2 \\
&\overset{(b)}{=} \int_0^{2\pi} \int_{\sqrt{t}}^{\infty} \frac{r}{2\pi(1 + \|c_i\|^2)} \exp\left(-\frac{r^2}{2(1 + \|c_i\|^2)}\right) \mathrm{d}\theta\mathrm{d}r \\
&= \int_{\sqrt{t}}^{\infty} \frac{r}{1 + \|c_i\|^2} \exp\left(-\frac{r^2}{2(1 + \|c_i\|^2)}\right) \mathrm{d}r \\
&= \exp\left(-\frac{t}{2(1 + \|c_i\|^2)}\right)
\end{aligned}$$

where,

- in (a), the region D is the region outside the yellow disk of radius $\sqrt{t}$ as shown in Fig. 1.
- in (b), we switch to polar coordinates in order to solve the integral.

(d) Since P_{e1}, P_{e2}, P_{e3} and P_{e4} are probabilities, then we know that $0 \leq P_{e1}, P_{e2}, P_{e3}, P_{e4} \leq 1$. Moreover, given that the decision rule used is the MAP decoder then the probability of error can't be bigger than $\frac{1}{2}$. This is due to the fact that the MAP rule minimizes the probability of error, and hence its corresponding probability of error (P_{MAP}) is less than or equal than the probability obtained with random decision (P_{RND}). Formally,

$$\begin{aligned} P_{MAP} &\leq P_{RND} \\ &= \frac{1}{2}\mathbb{P}(\text{error occured}|H=0) + \frac{1}{2}\mathbb{P}(\text{error occured}|H=1) \\ &= \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

Therefore, we get $0 \leq P_{e1}, P_{e2}, P_{e3}, P_{e4} \leq \frac{1}{2}$.

To order the above four probabilities, we will use the sufficient statistic $T = Y_1^2 + Y_2^2$ and the fact that the likelihood probabilities have exponential distributions: $f_{T|H}(t|i) = \frac{1}{2(1+\|c_i\|^2)} \exp\left(-\frac{t}{2(1+\|c_i\|^2)}\right)$. Moreovoer, since the priors are equiprobable then the MAP rule is equivalent to the ML rule.

- If $c_0 = (0,0)$ and $c_1 = (5,0)$, $f^{(1)}_{T|H}(t|0) = \frac{1}{2}\exp\left(-\frac{t}{2}\right) \neq f^{(1)}_{T|H}(t|1) = \frac{1}{52}\exp\left(-\frac{t}{52}\right)$. Hence, in this case $P_{e1} < \frac{1}{2}$.
- If $c_0 = (5,0)$ and $c_1 = (0,5)$, $f^{(2)}_{T|H}(t|0) = \frac{1}{52}\exp\left(-\frac{t}{52}\right) = f^{(2)}_{T|H}(t|1) = \frac{1}{52}\exp\left(-\frac{t}{52}\right)$. Hence, in this case $P_{e2} = \frac{1}{2}$.
- If $c_0 = (5,0)$ and $c_1 = (-5,0)$, $f^{(3)}_{T|H}(t|0) = \frac{1}{52}\exp\left(-\frac{t}{52}\right) = f^{(3)}_{T|H}(t|1) = \frac{1}{52}\exp\left(-\frac{t}{52}\right)$. Hence, in this case $P_{e3} = P_{e2} = \frac{1}{2}$.
- If $c_0 = (0,0)$ and $c_1 = (4,3)$, $f^{(4)}_{T|H}(t|0) = \frac{1}{2}\exp\left(-\frac{t}{2}\right) \neq f^{(4)}_{T|H}(t|1) = \frac{1}{52}\exp\left(-\frac{t}{52}\right)$. Hence, in this case $P_{e4} < \frac{1}{2}$. Moreover, we notice that $f^{(1)}_{T|H}(t|0) = f^{(4)}_{T|H}(t|0)$ and $f^{(1)}_{T|H}(t|1) = f^{(4)}_{T|H}(t|1)$. Therefore, in cases (1) and (4) we get the same decision rule and thus the same error probability. Hence, $P_{e4} = P_{e1}$.

Finally, $0 \leq P_{e1} = P_{e4} < P_{e2} = P_{e3} = \frac{1}{2}$.

Solution 2.

(a) The orthonormal basis is given in Fig. 2. From this figure, we see that $v_1(t) = v_0(t-T)$ and $\|v_0(t)\|^2 = \|v_1(t)\|^2 = 1$.

(b) The optimal receiver will use the matched filter $v_0(T-t)$ sampled at times T and $2T$ as seen in Fig. 3.

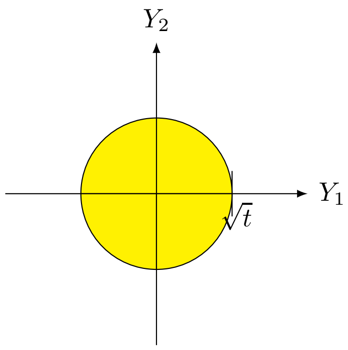


Figure 1: Plotting the integration region of question (c)

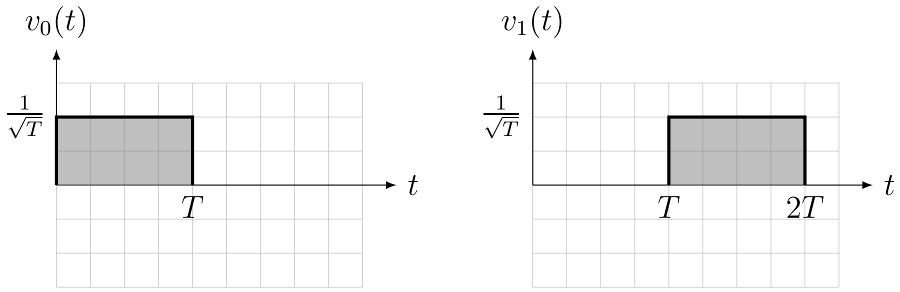


Figure 2: Orthonormal basis of Exercise 2-(a)

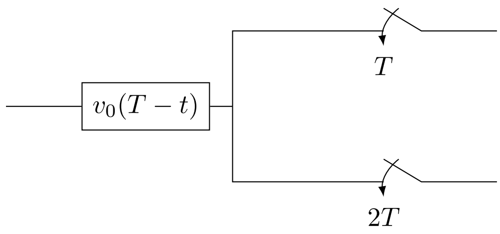
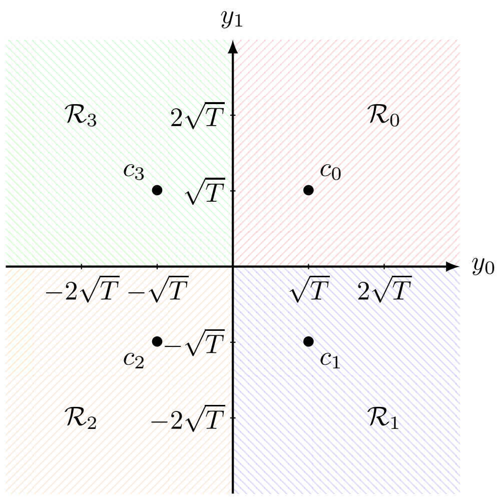


Figure 3: Block diagram for Exercise 2-(b)

(c)

$$
\begin{aligned}
w_0(t) &= \sqrt{T}v_0(t) + \sqrt{T}v_1(t) \\
w_1(t) &= \sqrt{T}v_0(t) - \sqrt{T}v_1(t) \\
w_2(t) &= -\sqrt{T}v_0(t) - \sqrt{T}v_1(t) \\
w_3(t) &= -\sqrt{T}v_0(t) + \sqrt{T}v_1(t)
\end{aligned}
$$

Since the priors are equiprobable and the channel is AWGN then we know from the course that the decision rule is minimum distance, as shown in the figure below.



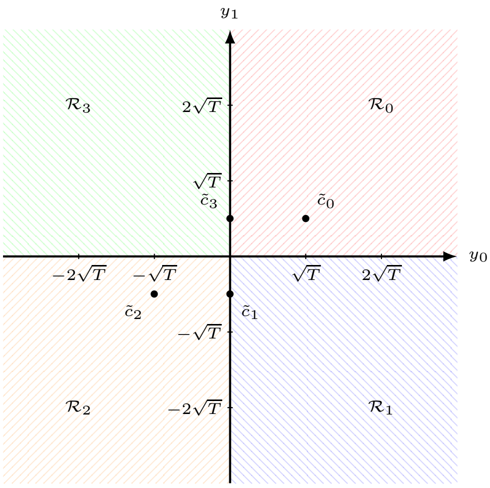
Hence the error probability is equal to the error probability given we have sent $w_0(t)$. Formally, denoting $Y_0^{(i)} = \langle w_i(t) + Z(t), v_0(t)\rangle$ and $Y_1^{(i)} = \langle w_i(t) + Z(t), v_1(t)\rangle$, where

$Z(t)$ is the white noise with spectral density $N_0/2$ and $i = 0, 1, 2, 3$, we have:

$$\begin{aligned} P_e = P_e(0) &= 1 - \mathbb{P}(\text{correct decision}|H=0) \\ &= 1 - \mathbb{P}(Y_0^{(0)} \geq 0, Y_1^{(0)} \geq 0) \\ &\overset{(a)}{=} 1 - \mathbb{P}(Y_0^{(0)} \geq 0)^2 \\ &\overset{(b)}{=} 1 - Q\left(-\sqrt{\frac{2T}{N_0}}\right)^2 \end{aligned}$$

where (a) is due to the fact that $Y_0^{(0)}$ and $Y_1^{(0)}$ are i.i.d, and (b) is due to the fact that $Y_0 \sim \mathcal{N}\left(\sqrt{T}, N_0/2\right)$.

(d) In this case the receiver keeps applying the decision rule in part (b) but the received signals are projected on delayed versions of the basis vectors, $\tilde{v}_0(t) = v_0\left(t - \frac{T}{2}\right)$ and $\tilde{v}_1(t) = v_1\left(t - \frac{T}{2}\right) = v_0\left(t - \frac{3T}{2}\right)$. The new constellation becomes

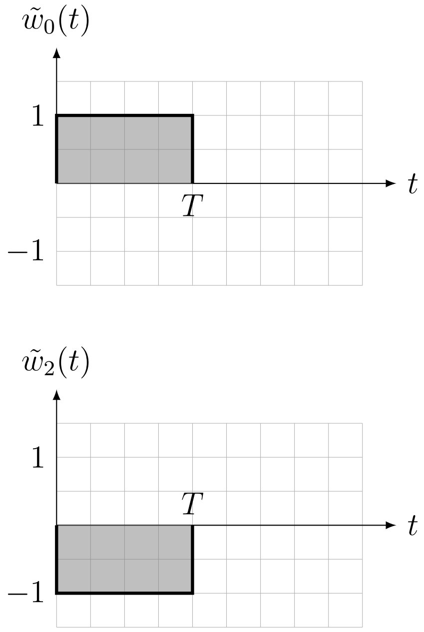


By symmetry we see that the error probabilities given $H = 0$ and $H = 2$ are equal ($P_e(0) = P_e(2)$) and that the error probabilities given $H = 1$ and $H = 3$ are also equal ($P_e(1) = P_e(3)$). Therefore, the error probability is given by:

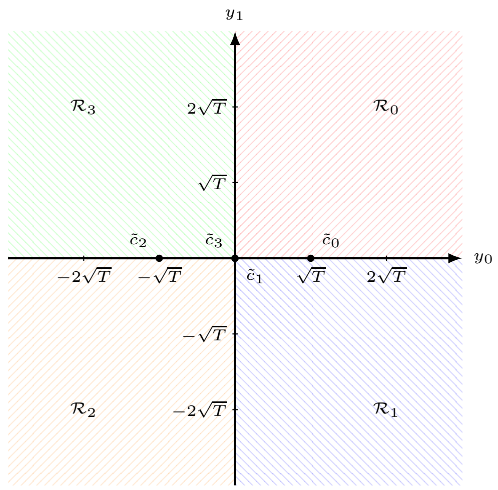
$$\begin{aligned} P_e &= \frac{1}{2}P_e(0) + \frac{1}{2}P_e(1) \\ &= \frac{1}{2}\left(1 - \mathbb{P}(\text{correct decision}|H=0)\right) + \frac{1}{2}\left(1 - \mathbb{P}(\text{correct decision}|H=1)\right) \\ &= \frac{1}{2}\left(1 - \mathbb{P}\left(Y_0^{(0)} \geq 0\right)\mathbb{P}\left(Y_1^{(0)} \geq 0\right)\right) + \frac{1}{2}\left(1 - \mathbb{P}\left(Y_0^{(1)} \geq 0\right)\mathbb{P}\left(Y_1^{(1)} \leq 0\right)\right) \\ &\overset{(a)}{=} \frac{1}{2}\left(1 - Q\left(-\sqrt{\frac{2T}{N_0}}\right)Q\left(-\sqrt{\frac{T}{2N_0}}\right)\right) + \frac{1}{2}\left(1 - Q(0)Q\left(-\sqrt{\frac{T}{2N_0}}\right)\right) \\ &= 1 - \frac{1}{2}Q\left(-\sqrt{\frac{T}{2N_0}}\right)\left(\frac{1}{2} + Q\left(-\sqrt{\frac{2T}{N_0}}\right)\right) \end{aligned}$$

where the equality in (a) is due to the fact that $Y_0^{(0)} \sim \mathcal{N}\left(\sqrt{T}, N_0/2\right)$, $Y_1^{(0)} \sim \mathcal{N}\left(\frac{\sqrt{T}}{2}, N_0/2\right)$, $Y_0^{(1)} \sim \mathcal{N}\left(0, N_0/2\right)$ and $Y_1^{(1)} \sim \mathcal{N}\left(-\frac{\sqrt{T}}{2}, N_0/2\right)$.

(e) In this case we notice that the suppot of $\tilde{w}_i(t)$ is in the interval $[0, T]$. The new transmitted waveforms are



Since the support of $v_1(t) = v_0(t - T)$ is in $[T, 2T]$, then the projection of the above waveforms on this vector is 0. Projecting the above waveforms on $v_0(t)$ we get the following constellation



By symmetry we see that the error probabilities given $H = 0$ and $H = 2$ are equal ($P_e(0) = P_e(2)$) and that the error probabilities given $H = 1$ and $H = 3$ are also equal

$(P_e(1) = P_e(3))$. Therefore, the error probability is given by:

$$\begin{aligned}
P_e &= \frac{1}{2}P_e(0) + \frac{1}{2}P_e(1) \\
&= \frac{1}{2}\left(1 - \mathbb{P}(\text{correct decision}|H=0)\right) + \frac{1}{2}\left(1 - \mathbb{P}(\text{correct decision}|H=1)\right) \\
&= \frac{1}{2}\left(1 - \mathbb{P}\left(Y_0^{(0)} \geq 0\right)\mathbb{P}\left(Y_1^{(0)} \geq 0\right)\right) + \frac{1}{2}\left(1 - \mathbb{P}\left(Y_0^{(1)} \geq 0\right)\mathbb{P}\left(Y_1^{(1)} \leq 0\right)\right) \\
&\overset{(a)}{=} \frac{1}{2}\left(1 - Q\left(-\sqrt{\frac{2T}{N_0}}\right)Q(0)\right) + \frac{1}{2}\left(1 - Q(0)Q(0)\right) \\
&= \frac{7}{8} - \frac{1}{4}Q\left(-\sqrt{\frac{2T}{N_0}}\right)
\end{aligned}$$

where the equality in (a) is due to the fact that $Y_0^{(0)} \sim \mathcal{N}\left(\sqrt{T}, N_0/2\right)$, $Y_1^{(0)} \sim \mathcal{N}(0, N_0/2)$, $Y_0^{(1)} \sim \mathcal{N}(0, N_0/2)$ and $Y_1^{(1)} \sim \mathcal{N}(0, N_0/2)$.

SOLUTION 3.

(a)

$$\begin{aligned}
\|\phi_1(t) + \phi_2(t) + \phi_3(t)\|^2 &= \langle \phi_1(t) + \phi_2(t) + \phi_3(t), \phi_1(t) + \phi_2(t) + \phi_3(t)\rangle \\
&= \langle \phi_1(t), \phi_1(t)\rangle + \langle \phi_1(t), \phi_2(t)\rangle + \langle \phi_1(t), \phi_3(t)\rangle \\
&\quad + \langle \phi_2(t), \phi_1(t)\rangle + \langle \phi_2(t), \phi_2(t)\rangle + \langle \phi_2(t), \phi_3(t)\rangle \\
&\quad + \langle \phi_3(t), \phi_1(t)\rangle + \langle \phi_3(t), \phi_2(t)\rangle + \langle \phi_3(t), \phi_3(t)\rangle \\
&\overset{(a)}{=} 3 - 6 \times \frac{1}{2} \\
&= 0
\end{aligned}$$

where (a) is due to the fact that $\|\phi_i\|^2 = 1$ and $\langle \phi_i, \phi_j\rangle = -\frac{1}{2}$ for $i, j = 1, 2, 3$ and $i \neq j$. The above equality implies that $\phi_1(t) + \phi_2(t) + \phi_3(t) = 0$ which means that the signals $\phi_1(t), \phi_2(t), \phi_3(t)$ are not linearly independent.

(b) Since $\phi_1(t), \phi_2(t), \phi_3(t)$ are linearly dependent, the space they span will have at most dimension 2. Now let us apply Gram–Schmidt procedure to find a basis for this span

$$\begin{aligned}
\psi_1(t) &= \phi_1(t) \\
\phi_2(t) &= \frac{\phi_2(t) - \langle \phi_2(t), \phi_1(t)\rangle \psi_1(t)}{\|\phi_2(t) - \langle \phi_2(t), \phi_1(t)\rangle \psi_1(t)\|} \\
&= \frac{\phi_2(t) + \frac{1}{2}\phi_1(t)}{\|\phi_2(t) + \frac{1}{2}\phi_1(t)\|}
\end{aligned}$$

Now,

$$\begin{aligned}
\|\phi_2(t) + \frac{1}{2}\phi_1(t)\|^2 &= \langle \phi_2(t), \frac{1}{2}\phi_1(t)\rangle + \langle \phi_2(t), \phi_2(t)\rangle + \frac{1}{2}\langle \phi_1(t), \phi_2(t)\rangle + \frac{1}{4}\langle \phi_1(t), \phi_1(t)\rangle \\
&= \frac{3}{4} \neq 0
\end{aligned}$$

So $\psi_2(t) = \frac{2}{\sqrt{3}}\phi_2(t) + \frac{1}{\sqrt{3}}\phi_1(t) \neq 0$.
Hence the dimension $n = 2$.

(c) We know that

$$\begin{aligned}
\phi_1(t) &= \psi_1(t)\\
\phi_2(t) &= \frac{\sqrt{3}}{2}\psi_2(t) - \frac{1}{2}\psi_1(t)\\
\phi_3(t) &= -\phi_1(t) - \phi_2(t)\\
&= -\psi_1(t) - \frac{\sqrt{3}}{2}\psi_2(t) + \frac{1}{2}\psi_1(t)\\
&= -\frac{1}{2}\psi_1(t) - \frac{\sqrt{3}}{2}\psi_2(t)
\end{aligned}$$

Hence,

$$\begin{aligned}
w_i(t) &= \sum_{j=1}^{3} c_{ij}\phi_j(t)\\
&= c_{i1}\phi_1 + c_{i2}\phi_2 + c_{i3}\phi_3\\
&= c_{i1}\psi_1(t) + c_{i2}\frac{\sqrt{3}}{2}\psi_2(t) - \frac{c_{i2}}{2}\psi_1(t) - \frac{c_{i3}}{2}\psi_1(t) - \frac{\sqrt{3}}{2}c_{i3}\psi_2(t)\\
&= \psi_1(t)\left(c_{i1} - \frac{c_{i2}}{2} - \frac{c_{i3}}{2}\right) + \psi_2(t)\left(\frac{\sqrt{3}}{2}(c_{i2} - c_{i3})\right)
\end{aligned}$$

So $\tilde{c}_i = \left(\left(c_{i1} - \frac{c_{i2}}{2} - \frac{c_{i3}}{2}\right), \frac{\sqrt{3}}{2}(c_{i2} - c_{i3})\right)$ and

$$\begin{aligned}
\tilde{c}_0 &= (0, 0)\\
\tilde{c}_1 &= (-3, \sqrt{3})\\
\tilde{c}_2 &= (-3, -\sqrt{3})
\end{aligned}$$

(d) Assuming the waveforms are chosen uniformly at random, the mean of the constellation found in (c) is given by $m = \frac{1}{3}\tilde{c}_0 + \frac{1}{3}\tilde{c}_1 + \frac{1}{3}\tilde{c}_2 = (-2, 0)$. Hence the new constellation becomes

$$\begin{aligned}
\hat{c}_0 &= \tilde{c}_0 - m = (2, 0)\\
\hat{c}_1 &= \tilde{c}_1 - m = (-1, \sqrt{3})\\
\hat{c}_2 &= \tilde{c}_2 - m = (-1, -\sqrt{3})
\end{aligned}$$

(e)

$$
\begin{aligned}
f_0(y)f_1(y) &= \frac{1}{(2\pi\sigma^2)^{\frac{d}{2}}} \exp\left(-\frac{\|y-\mu_0\|^2}{2\sigma^2}\right) \frac{1}{(2\pi\sigma^2)^{\frac{d}{2}}} \exp\left(-\frac{\|y-\mu_1\|^2}{2\sigma^2}\right) \\
&= \left(\frac{1}{(2\pi\sigma^2)^{\frac{d}{2}}}\right)^2 \exp\left(-\frac{1}{2\sigma^2}\left(\|y-\mu_0\|^2 + \|y-\mu_1\|^2\right)\right) \\
&= \left(\frac{1}{(2\pi\sigma^2)^{\frac{d}{2}}}\right)^2 \exp\left(-\frac{1}{2\sigma^2}\left(y^Ty - 2y^T\mu_0 + \mu_0^T\mu_0 + y^Ty - 2y^T\mu_1 + \mu_1^T\mu_1\right)\right) \\
&= \left(\frac{1}{(2\pi\sigma^2)^{\frac{d}{2}}}\right)^2 \exp\left(-\frac{1}{2\sigma^2}\left(2\|y\|^2 - 2y^T(\mu_0+\mu_1) + \|\mu_0\|^2 + \|\mu_1\|^2\right)\right) \\
&= \left(\frac{1}{(2\pi\sigma^2)^{\frac{d}{2}}}\right)^2 \left[\exp\left(-\frac{1}{2\sigma^2}\left(\|y - \frac{\mu_0+\mu_1}{2}\|^2\right)\right)\right]^2 \\
&\times \exp\left(-\frac{1}{2\sigma^2}\left(\|\mu_0\|^2 + \|\mu_1\|^2 - \frac{\|\mu_0+\mu_1\|^2}{2}\right)\right) \\
&= f(y)^2 \exp\left(-\frac{1}{2\sigma^2}\left(\|\mu_0\|^2 + \|\mu_1\|^2 - \frac{\|\mu_0+\mu_1\|^2}{2}\right)\right) \\
&= f(y)^2 \exp\left(-\frac{1}{2\sigma^2}\left(\frac{1}{2}\|\mu_0\|^2 + \frac{1}{2}\|\mu_1\|^2 - \mu_0^T\mu_1\right)\right) \\
&= f(y)^2 \exp\left(-\frac{\|\mu_0-\mu_1\|^2}{4\sigma^2}\right)
\end{aligned}
$$

where d is the dimension of the random vectors.

(f) First notice that we have three types of likelihood probabilities: $f_{Y|H}(y|0) = f_0(y)$ which is the density function of the Gaussian random vector $\mathcal{N}\left(\hat{c}_0, \frac{1}{2}\right)$, $f_{Y|H}(y|1) = f_1(y)$ which is the density function of the Gaussian random vector $\mathcal{N}\left(\hat{c}_1, \frac{1}{2}\right)$, $f_{Y|H}(y|2) = f_2(y)$ which is the density function of the Gaussian random vector $\mathcal{N}\left(\hat{c}_2, \frac{1}{2}\right)$. Hence, applying our previous result we get:

$$
\begin{aligned}
f_0(y)f_1(y) &= f_{Y_0}(y)^2 \exp\left(-\frac{\|\hat{c}_0 - \hat{c}_1\|^2}{2}\right) \\
&= f_{Y_0}(y)^2 \exp\left(-\frac{12}{2}\right) \\
&= f_{Y_0}(y)^2 \exp(-6)
\end{aligned}
$$

where $Y_0 \sim \mathcal{N}\left(\frac{\hat{c}_0+\hat{c}_1}{2}, \frac{1}{2}\right)$. Similarly,

$$
\begin{aligned}
f_0(y)f_2(y) &= f_{Y_1}(y)^2 \exp\left(-\frac{\|\hat{c}_0 - \hat{c}_2\|^2}{2}\right) \\
&= f_{Y_1}(y)^2 \exp(-6) \\
f_1(y)f_2(y) &= f_{Y_2}(y)^2 \exp\left(-\frac{\|\hat{c}_1 - \hat{c}_2\|^2}{2}\right) \\
&= f_{Y_2}(y)^2 \exp(-6)
\end{aligned}
$$

where $Y_1 \sim \mathcal{N}\left(\frac{\hat{c}_0+\hat{c}_2}{2}, \frac{1}{2}\right)$ and $Y_2 \sim \mathcal{N}\left(\frac{\hat{c}_2+\hat{c}_1}{2}, \frac{1}{2}\right)$.

Assuming the priors are equiprobable and using the Bhattacharyya bound, the error probability is bounded by:

$$\begin{aligned}
P_e &\leq \frac{1}{3}\left(\int_{y\in\mathbb{R}^2}\sqrt{f_0(y)f_1(y)}\mathrm{d}y + \int_{y\in\mathbb{R}^2}\sqrt{f_0(y)f_2(y)}\mathrm{d}y + \int_{y\in\mathbb{R}^2}\sqrt{f_1(y)f_0(y)}\mathrm{d}y\right. \\
&\quad \left. + \int_{y\in\mathbb{R}^2}\sqrt{f_1(y)f_2(y)}\mathrm{d}y + \int_{y\in\mathbb{R}^2}\sqrt{f_2(y)f_0(y)}\mathrm{d}y + \int_{y\in\mathbb{R}^2}\sqrt{f_2(y)f_1(y)}\mathrm{d}y\right) \\
&= \frac{2}{3}\left(\int_{y\in\mathbb{R}^2}\sqrt{f_0(y)f_1(y)}\mathrm{d}y + \int_{y\in\mathbb{R}^2}\sqrt{f_0(y)f_2(y)}\mathrm{d}y + \int_{y\in\mathbb{R}^2}\sqrt{f_2(y)f_1(y)}\mathrm{d}y\right) \\
&= \frac{2}{3}\left(\int_{y\in\mathbb{R}^2} f_{Y_0}(y)\exp\left(-3\right)\mathrm{d}y + \int_{y\in\mathbb{R}^2} f_{Y_1}(y)\exp\left(-3\right)\mathrm{d}y + \int_{y\in\mathbb{R}^2} f_{Y_2}(y)\exp\left(-3\right)\mathrm{d}y\right) \\
&= \frac{2}{3}\left(3\exp(-3)\right) \\
&= 2\exp(-3)
\end{aligned}$$