

SOLUTION 1.

(a)

$$\begin{aligned} R_\xi(\tau) &= \int_{-\infty}^{\infty} \xi(t+\tau)\xi^*(t) dt = \langle \xi(t+\tau), \xi(t) \rangle \\ &\stackrel{(1)}{\leq} \|\xi(t+\tau)\| \cdot \|\xi(t)\| = \|\xi\| \cdot \|\xi\| = \|\xi\|^2 \stackrel{(2)}{=} R_\xi(0), \end{aligned}$$

where (1) follows from the Cauchy–Schwarz inequality and (2) from the fact that  $R_\xi(0) = \int_{-\infty}^{\infty} \xi(t)\xi^*(t) dt = \|\xi\|^2$ .

(b)

$$\begin{aligned} R_\xi(-\tau) &= \int_{-\infty}^{\infty} \xi(t-\tau)\xi^*(t) dt \\ &= \left( \int_{-\infty}^{\infty} \xi(t)\xi^*(t-\tau) dt \right)^* \\ &\stackrel{t \rightarrow t+\tau}{=} R_\xi^*(\tau). \end{aligned}$$

(c)

$$\begin{aligned} R_\xi(\tau) &= \int_{-\infty}^{\infty} \xi(t+\tau)\xi^*(t) dt \\ &\stackrel{t \rightarrow t-\tau}{=} \int_{-\infty}^{\infty} \xi(t)\xi^*(t-\tau) dt \\ &= \xi(\tau) \star \xi^*(-\tau). \end{aligned}$$

(d) By Parseval's identity, we have

$$\begin{aligned} R_\xi(\tau) &= \langle \xi(t+\tau), \xi(t) \rangle \\ &= \langle \xi_{\mathcal{F}}(f)e^{j2\pi f\tau}, \xi_{\mathcal{F}}(f) \rangle \\ &= \int_{-\infty}^{\infty} \xi_{\mathcal{F}}(f)\xi_{\mathcal{F}}^*(f)e^{j2\pi f\tau} df \\ &= \int_{-\infty}^{\infty} |\xi_{\mathcal{F}}(f)|^2 e^{j2\pi f\tau} df, \end{aligned}$$

which is the inverse Fourier transform of  $|\xi_{\mathcal{F}}(f)|^2$ .

SOLUTION 2.

(a) We have

$$y(t) = \int_{-\infty}^{\infty} w(\tau)\psi(\tau - t)d\tau.$$

The samples of this waveform at multiples of  $T$  are

$$\begin{aligned} y(mT) &= \int_{-\infty}^{\infty} w(\tau)\psi(\tau - mT)d\tau \\ &= \int_{-\infty}^{\infty} \left[ \sum_{k=1}^K d_k \psi(\tau - kT) \right] \psi(\tau - mT)d\tau \\ &= \sum_{k=1}^K d_k \int_{-\infty}^{\infty} \psi(\tau - kT)\psi(\tau - mT)d\tau \\ &= \sum_{k=1}^K d_k \mathbb{1}\{k = m\} \\ &= d_m. \end{aligned}$$

(b) Let  $\tilde{w}(t)$  be the channel output. Then,  $\tilde{y}(t)$  is  $\tilde{w}(t)$  filtered by  $\psi(-t)$ . We have

$$\tilde{w}(t) = w(t) + \rho w(t - T)$$

and

$$\tilde{y}(t) = \int_{-\infty}^{\infty} \tilde{w}(\tau)\psi(\tau - t)d\tau.$$

The samples of this waveform at multiples of  $T$  are

$$\begin{aligned} \tilde{y}(mT) &= \int_{-\infty}^{\infty} \tilde{w}(\tau)\psi(\tau - mT)d\tau \\ &= \int_{-\infty}^{\infty} [w(\tau) + \rho w(\tau - T)]\psi(\tau - mT)d\tau \\ &= \int_{-\infty}^{\infty} \left[ \sum_{k=1}^K d_k \psi(\tau - kT) \right] \psi(\tau - mT)d\tau + \\ &\quad \rho \int_{-\infty}^{\infty} \left[ \sum_{k=1}^K d_k \psi(\tau - T - kT) \right] \psi(\tau - mT)d\tau \\ &= \sum_{k=1}^K d_k \mathbb{1}\{k = m\} + \rho \sum_{k=1}^K d_k \mathbb{1}\{k = m - 1\} \\ &= d_m + \rho d_{m-1}. \end{aligned}$$

(c) From the symmetry of the problem, we have

$$P_e = P_e(1) = P_e(-1).$$

$$\begin{aligned}
P_e(1) &= \Pr\{\hat{D}_k = -1 | D_k = 1, D_{k-1} = -1\} \Pr\{D_{k-1} = -1\} + \\
&\quad \Pr\{\hat{D}_k = -1 | D_k = 1, D_{k-1} = 1\} \Pr\{D_{k-1} = 1\} \\
&= \frac{1}{2} (\Pr\{Y_k < 0 | D_k = 1, D_{k-1} = -1\} + \Pr\{Y_k < 0 | D_k = 1, D_{k-1} = 1\}) \\
&= \frac{1}{2} (\Pr\{1 - \alpha + Z_k < 0\} + \Pr\{1 + \alpha + Z_k < 0\}) \\
&= \frac{1}{2} (\Pr\{Z_k < -1 + \alpha\} + \Pr\{Z_k < -1 - \alpha\}) \\
&= \frac{1}{2} \left[ Q\left(\frac{1 - \alpha}{\sigma}\right) + Q\left(\frac{1 + \alpha}{\sigma}\right) \right].
\end{aligned}$$

SOLUTION 3.

(a) We can easily see that

$$\mathbb{E}[X_i | X_{i-1}] = \frac{1}{2}X_{i-1} + \frac{1}{2}(-X_{i-1}) = 0.$$

Consequently (using the law of total expectation)

$$\mathbb{E}[X_i] = \mathbb{E}[\mathbb{E}[X_i | X_{i-1}]] = 0.$$

Therefore,

$$K_X[k] = \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_{i-k} - \mathbb{E}[X_{i-k}])^*] = \mathbb{E}[X_i X_{i-k}^*]$$

Moreover, using the fact that  $X_i = X_{i-1} \times (-1)^{D_i}$  repeatedly, we can write

$$X_i = X_{i-k} \times \prod_{j=i-k+1}^i (-1)^{D_j}$$

Thus,

$$\begin{aligned}
K_X[k] &= \mathbb{E}[X_i X_{i-k}^*] \\
&= \mathbb{E} \left[ X_{i-k} \prod_{j=i-k+1}^i (-1)^{D_j} X_{i-k}^* \right] \\
&\stackrel{(a)}{=} \mathbb{E}[X_{i-k} X_{i-k}^*] \prod_{j=i-k+1}^i \mathbb{E}[(-1)^{D_j}] \\
&= \mathcal{E} \prod_{j=i-k+1}^i \mathbb{E}[(-1)^{D_j}] \\
&\stackrel{(b)}{=} \begin{cases} \mathcal{E} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

where (a) follows from the independence of data bits  $\{D_i\}$  and (b) since  $\mathbb{E}[(-1)^{D_i}] = 0$ .

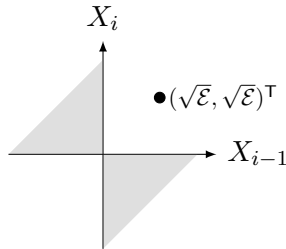
(b) By sampling the signal at the output of the matched filter,  $Y(t)$ , at multiples of  $T$ , we obtain

$$Y(iT) = X_i + Z_i,$$

where  $Z_i$  is normally distributed with zero mean and variance  $N_0/2$ . By looking at the definition of  $X_i$ , we see that it is equal to  $X_{i-1}$  if  $D_i = 0$  and equal to  $-X_{i-1}$  if  $D_i = 1$ . Therefore a simple decoder estimates that  $\hat{D}_i = 0$  if  $Y_i$  and  $Y_{i-1}$  have the same sign, and  $\hat{D}_i = 1$  otherwise. This is equivalent to

$$Y_i Y_{i-1} \underset{\hat{D}_i=1}{\overset{\hat{D}_i=0}{\gtrless}} 0.$$

- (c) We first compute the error probability when  $D_i = 0$ . If  $X_{i-1} = \sqrt{\mathcal{E}}$ , then  $X_i = \sqrt{\mathcal{E}}$ . When we decode, we will make an error if the signal  $(Y_{i-1}, Y_i)^\top$  is in the second or fourth quadrants (shaded regions in the following figure).



Due to the symmetry of the problem, the probability for this to happen is two times the probability for  $(Y_{i-1}, Y_i)^\top$  to be in the second quadrant:

$$\Pr\{Z_{i-1} < -\sqrt{\mathcal{E}} \cap Z_i > -\sqrt{\mathcal{E}}\} = Q\left(\sqrt{\frac{\mathcal{E}}{N_0/2}}\right) Q\left(-\sqrt{\frac{\mathcal{E}}{N_0/2}}\right),$$

so,

$$P_e(D_i = 0 | D_{i-1} = 0) = 2Q\left(\sqrt{\frac{\mathcal{E}}{N_0/2}}\right) Q\left(-\sqrt{\frac{\mathcal{E}}{N_0/2}}\right).$$

Again, due to the symmetry of the problem,

$$P_e(D_i = 0 | D_{i-1} = 1) = P_e(D_i = 0 | D_{i-1} = 0) = P_e(D_i = 0),$$

and

$$P_e(D_i = 1) = P_e(D_i = 0);$$

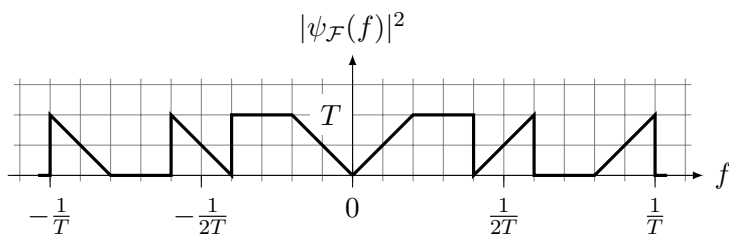
hence

$$P_e = 2Q\left(\sqrt{\frac{\mathcal{E}}{N_0/2}}\right) Q\left(-\sqrt{\frac{\mathcal{E}}{N_0/2}}\right).$$

**SOLUTION 4.** Because  $\psi(t)$  is real, its Fourier transform is conjugate symmetric ( $\psi_{\mathcal{F}}(f) = \psi_{\mathcal{F}}^*(-f)$ ).

From the condition  $\int \psi(t - kT)\psi(t - lT)dt = \mathbb{1}\{k = l\}$  for every pair  $k, l$ , it follows that  $|\psi_{\mathcal{F}}(f)|^2$  satisfies Nyquist's criterion with parameter  $T$ ,  $\sum_{k \in \mathbb{Z}} |\psi_{\mathcal{F}}(f - k/T)|^2 = T$ . On the other hand, since  $\psi_{\mathcal{F}}(f) = 0$  for  $|f| > \frac{1}{T}$ ,  $|\psi_{\mathcal{F}}(f)|^2$  must have band-edge symmetry.

Putting everything together, we obtain the complete plot of  $|\psi_{\mathcal{F}}(f)|^2$ .



SOLUTION 5. From Theorem 5.6, we know that  $\{\psi(t - jT)\}_{j=-\infty}^{\infty}$  is an orthonormal set if and only if

$$\sum_{k \in \mathbb{Z}} |\psi_{\mathcal{F}}(f - \frac{k}{T})|^2 = T.$$

(a)

$$\sum_{k \in \mathbb{Z}} T \mathbb{1}_{[\frac{k}{T} - \frac{1}{2T}, \frac{k}{T} + \frac{1}{2T}]}(f) = T \Rightarrow \text{The Nyquist criterion is satisfied}$$

$\Rightarrow \psi(t)$  is orthonormal to its time-translates by multiples of  $T$ .

(b)

$$\sum_{k \in \mathbb{Z}} \frac{T}{2} \mathbb{1}_{[\frac{k-1}{T}, \frac{k+1}{T}]}(f) = T \Rightarrow \text{The Nyquist criterion is satisfied}$$

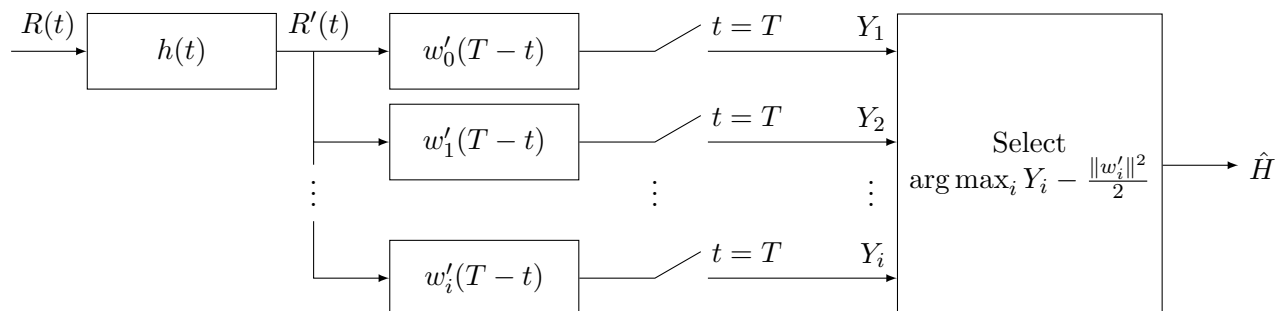
$\Rightarrow \psi(t)$  is orthonormal to its time-translates by multiples of  $T$ .

(c) Because  $|\psi_{\mathcal{F}}(f)|^2$  vanishes outside  $[-\frac{1}{T}, \frac{1}{T}]$ , we verify whether the band-edge symmetry is fulfilled, which is the case. Hence, the Nyquist criterion is satisfied and  $\psi(t)$  is orthonormal to its time-translates by multiples of  $T$ . Note: the same reasoning can be applied to (b).

(d)  $\psi_{\mathcal{F}}(f)$  is a sinc function, therefore  $\psi(t)$  is a box function, equal to  $\frac{1}{T} \mathbb{1}_{[-\frac{T}{2}, \frac{T}{2}]}(t)$ . This is orthogonal to its time-translates by multiples of  $T$ , but does not have unit norm (unless  $T = 1$ ):  $\int_{-\infty}^{\infty} |\psi(t)|^2 dt = \frac{1}{T}$ .

SOLUTION 6.

(a) We pass  $R(t)$  through a whitening filter  $h(t)$  such that the output  $R'(t)$  looks like the output of an AWGN channel. After this step we are facing a familiar situation and can implement a matched filter receiver. The receiver architecture is shown below:



Let  $N'(t) = \int N(\alpha)h(t - \alpha) d\alpha$  be the noise at the output of the whitening filter. We want to select the filter  $h(t)$  such that  $\frac{N_0}{2} = G(f)|h_{\mathcal{F}}(f)|^2$ , i.e.,

$$|h_{\mathcal{F}}(f)|^2 = \frac{N_0}{2G(f)}.$$

The output of the filter is

$$\begin{aligned} R'(t) &= \int R(\alpha)h(t - \alpha) d\alpha = \int w_i(\alpha)h(t - \alpha) d\alpha + \int N(\alpha)h(t - \alpha) d\alpha \\ &= w'_i(t) + N'(t), \end{aligned}$$

where  $N'(t)$  is white Gaussian noise and  $w'_i(t) = \int w_i(\alpha)h(t - \alpha) d\alpha$ . We need to design the matched filter for the signals  $w'_i(t)$ .

- (b) To minimize both the noise and the energy of the signal, we need to select an antipodal signal pair that is frequency-limited to  $[a, b]$  and has energy  $\mathcal{E}$ .