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Handout 17 Principles of Digital Communications
Solutions to Problem Set 7 Apr. 07, 2017

SOLUTION 1.

(a) The Cauchy–Schwarz inequality states

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$$

with equality if and only if $x = \alpha y$ for some scalar α. For our problem, we can write

$$|\langle w, \phi\rangle|^2 \leq \|w\|^2 \cdot \|\phi\|^2 = \|w\|^2$$

with equality if and only if $\phi = \alpha w$ for some scalar α. Thus, the maximizing $\phi(t)$ is simply a scaled version of $w(t)$.

REMARK. *In two dimensions, we have $|\langle x, y\rangle| = \|x\| \cdot \|y\| \cos\alpha$, where α is the angle between the two vectors. It is clear that the maximum is achieved when $\cos\alpha = 1 \Leftrightarrow \alpha = 0$ (or $\alpha = k2\pi$). Thus, x and y are colinear.*

(b) The problem is

$$\max_{\phi_1,\phi_2} (c_1\phi_1 + c_2\phi_2) \text{ subject to } \phi_1^2 + \phi_2^2 = 1$$

Thus, we can reduce by setting $\phi_2 = \sqrt{1-\phi_1^2}$ to obtain

$$\max_{\phi_1} \left(c_1\phi_1 + c_2\sqrt{1-\phi_1^2} \right)$$

This maximum is found by taking the derivative:

$$\frac{d}{d\phi_1} \left(c_1\phi_1 + c_2\sqrt{1-\phi_1^2} \right) = c_1 - c_2 \frac{\phi_1}{\sqrt{1-\phi_1^2}}$$

Setting this equal to zero yields $c_1 = c_2 \frac{\phi_1}{\sqrt{1-\phi_1^2}}$, i.e,

$$c_1^2 = c_2^2 \frac{\phi_1^2}{1-\phi_1^2}$$

This immediately gives $\phi_1 = \frac{c_1}{\sqrt{c_1^2+c_2^2}}$ and thus $\phi_2 = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$, which are colinear to c_1 and c_2 respectively.

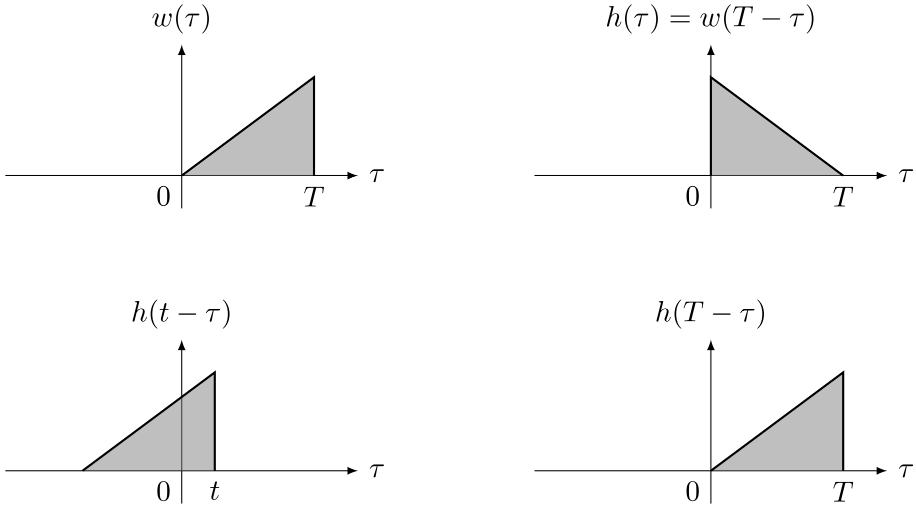
Note: the goal of this exercise was to display yet another way to derive the matched filter.

(c) Passing an input $w(t)$ through a filter with impulse response $h(t)$ generates output waveform $y(t) = \int w(\tau)h(t-\tau)d\tau$. If this waveform $y(t)$ is sampled at time $t = T$, then the output sample is

$$y(T) = \int w(\tau)h(T-\tau)d\tau \tag{1}$$

An example signal $w(\tau)$ is shown below (top left). The filter is then the waveform shown on the top right, and the convolution term of the filter on the bottom left. Finally, the filter term $h(T-\tau)$ of Equation (1) is shown on the bottom right. One can see that $h(T-\tau) = w(\tau)$, so indeed

$$y(T) = \int w(\tau)h(T-\tau)d\tau = \int w^2(\tau)d\tau = \int_0^T w^2(\tau)d\tau$$



Solution 2.

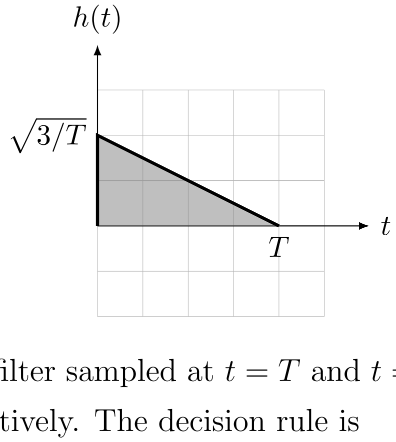
(a) The binary hypothesis testing problem may be written as:

$$\begin{aligned} H = 0: &\quad R(t) = w_1(t) + N(t) \\ H = 1: &\quad R(t) = w_2(t) + N(t) \end{aligned}$$

The impulse response of a matched filter is

$$h(t) = \frac{w_1(T-t)}{\|w_1(t)\|}$$

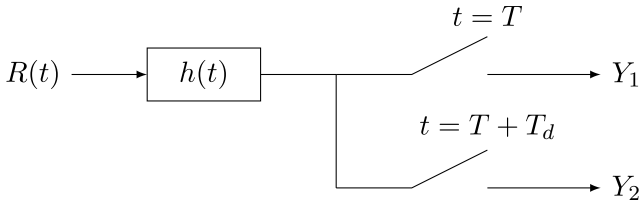
and is shown below. We have normalized the impulse response of the matched filter to have unit norm. Note that this does not affect the probability of error.



The output of the matched filter sampled at $t = T$ and $t = T + T_d$ is $Y_1 = \langle R(t), \frac{w_1(t)}{\|w_1\|}\rangle$ and $Y_2 = \langle R(t), \frac{w_2(t)}{\|w_2\|}\rangle$ respectively. The decision rule is

$$Y_1 \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} Y_2$$

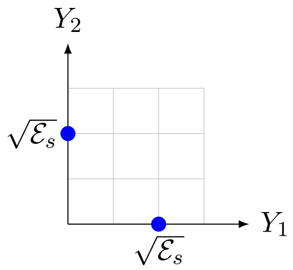
The block diagram of the system is shown below.



(b) For $T_d \geq T$, the signals $w_1(t)$ and $w_2(t)$ are orthogonal to each other. Let

$$\mathcal{E}_s = \|w_1\|^2 = \frac{A^2 T}{3}$$

(The signal space representation of the constellation can be seen below.)



The noise $Z_1, Z_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$ and Z_1 is independent of Z_2. The probability of error can be readily calculated as

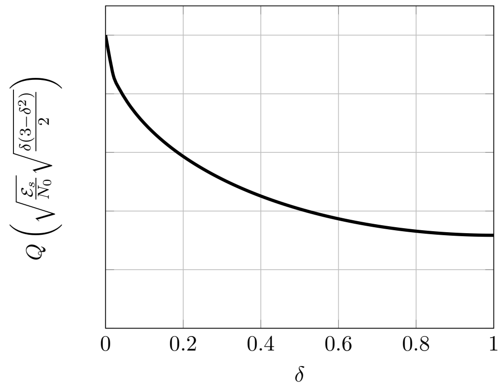
$$P_e = Q\left(\frac{\sqrt{2\mathcal{E}_s}}{2\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{\mathcal{E}_s}{N_0}}\right)$$

For $T_d \leq T$ we note that (since the receiver has not changed) still $(Z_1, Z_2) \sim \mathcal{N}(0, \frac{N_0}{2} I)$ and, hence, the error probability equals $Q(\frac{d}{2\sqrt{N_0/2}})$ where d is the distance between two codewords. Therefore, we compute

$$\begin{aligned}
\|w_1(t) - w_2(t)\|^2 &= \int (w_1(t) - w_2(t))^2 dt \\
&= \int_0^{T_d} \left(\frac{A}{T}\right)^2 t^2 dt + \int_{T_d}^{T} \left(T_d \frac{A}{T}\right)^2 dt + \int_T^{T+T_d} \left(\frac{A}{T}\right)^2 (t - T_d)^2 dt \\
&= \left(\frac{A}{T}\right)^2 \left[\frac{T_d^3}{3} + T_d^2 (T - T_d) + \frac{T^3 - (T - T_d)^3}{3}\right] \\
&\overset{(\star)}{=} \left(\frac{A}{T}\right)^2 \frac{1}{3} T^3 \delta (3 - \delta^2) \\
&= \mathcal{E}_s \delta (3 - \delta^2)
\end{aligned}$$

where in $(\star)$ we have defined $\delta = \frac{T_d}{T}$. Given this, we can compute

$$P_e = Q\left(\sqrt{\frac{\mathcal{E}_s}{N_0}} \sqrt{\frac{\delta(3 - \delta^2)}{2}}\right)$$

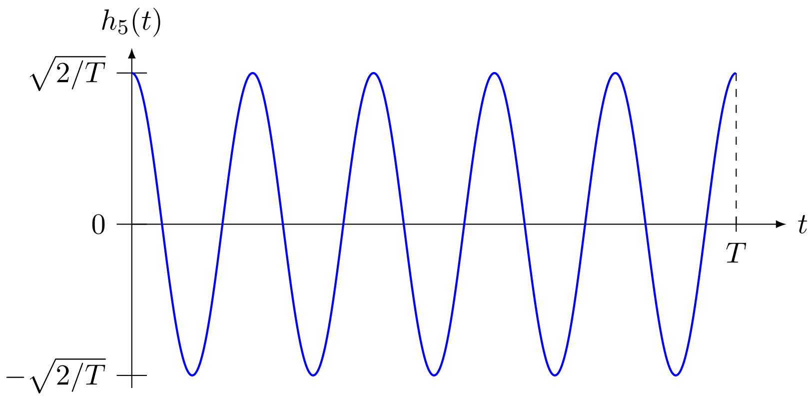


SOLUTION 3.

(a) The matched filter is the filter whose impulse response is a delayed, time-reversed version of $w_j(t)$, i.e.

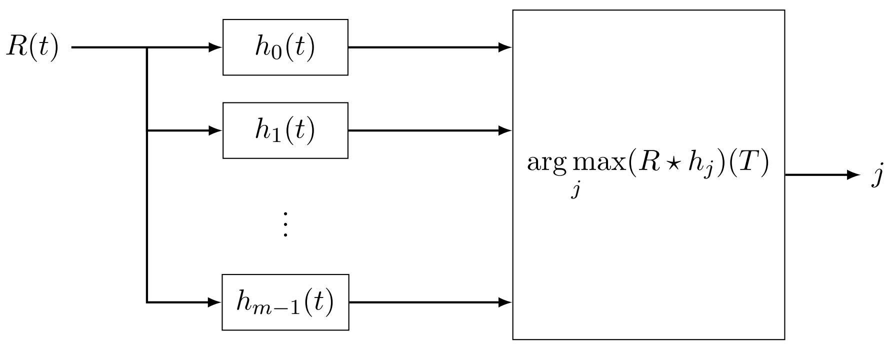
$$\begin{aligned} h_j(t) &= w_j(T-t) = \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi n_j (T-t)}{T}\right) \mathbb{1}_{[0,T]}(T-t) \\ &= \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi n_j t}{T}\right) \mathbb{1}_{[0,T]}(t) \end{aligned}$$

As an example, $h_5(t)$ is shown below.



The receiver then processes the received signal $R(t)$ through the matched filter $h_j(t)$ to obtain $(R \star h_j)(t)$. This signal is sampled at time T to yield the value needed for the MAP decision.

(b) We need m matched filters, one for each signal.

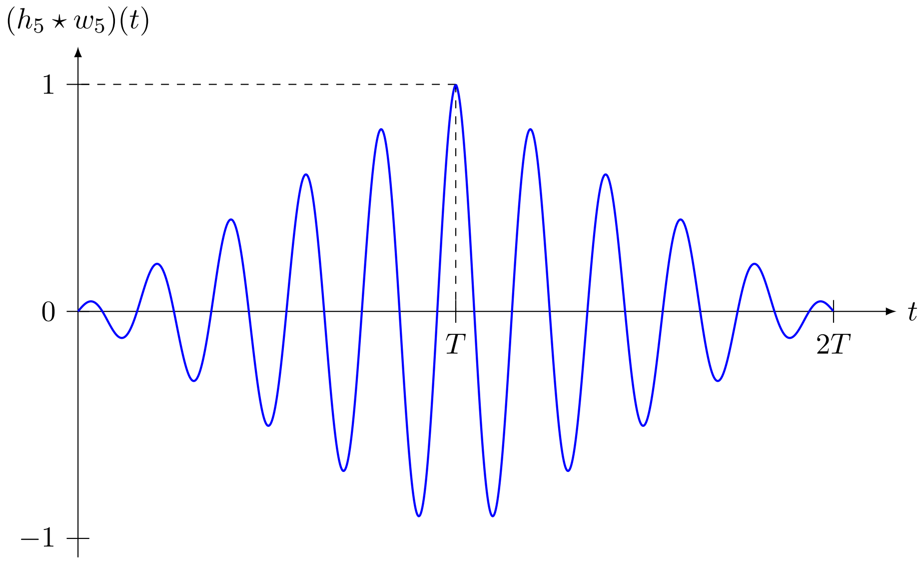


(c) The following **matlab** program computes the output of the matched filter $h_5(t)$.

```
T = 1;
Resolution = 1e-3;
t = 0:Resolution:T;
nj = 5;

wj = sqrt(2/T) * cos ( (2*pi*nj*t)/T );
hj = sqrt(2/T) * cos ( (2*pi*nj*t)/T );

output = conv(wj, hj);
```



Note that the resulting signal is *zero* for $t \leq 0$ and also for $t \geq 2T$. The figure also reveals why sampling at time $t = T$ is a good idea: the value of the matched filter output signal is maximal.

Solution 4.

(a) The third component of c_i is zero for all i. Furthermore Z_1, Z_2 and Z_3 are zero mean i.i.d. Gaussian random variables. Hence,

$$f_{Y|H}(y|i) = f_{Z_1}(y_1 - c_{i,1}) f_{Z_2}(y_2 - c_{i,2}) f_{Z_3}(y_3),$$

which is in the form $g_i(T(y))h(y)$ for $T(y) = (y_1, y_2)^\mathsf{T}$ and $h(y) = f_{Z_3}(y_3)$. Hence, by the Fisher–Neyman factorization theorem, $T(Y) = (Y_1, Y_2)^\mathsf{T}$ is a sufficient statistic.

(b) We have $Y_3 = Z_3 = Z_2$. By observing Y_3, we can remove the noise in the second component of Y. Specifically, we have $c_{i,2} = Y_2 - Y_3$. If the second component is different for each hypothesis, then the receiver can make an error-free decision which is not possible using only $(Y_1, Y_2)^\mathsf{T}$ (see the next question for more on this). We can see that Y_3 contains very useful information and can't be discarded. Therefore, $(Y_1, Y_2)^\mathsf{T}$ is not a sufficient statistic.

(c) If we have only $(Y_1, Y_2)^\mathsf{T}$ then the hypothesis testing problem will be

$$H = i : (Y_1, Y_2) = (c_{i,1}, c_{i,2}) + (Z_1, Z_2) \quad i = \{0, 1\}$$

Using the fact that $c_0 = (1, 0, 0)^\mathsf{T}$ and $c_1 = (0, 1, 0)^\mathsf{T}$, the ML test becomes

$$y_1 - y_2 \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0$$

Under $H = 0$, $Y_1 - Y_2$ is a Gaussian random variable with mean 1 and variance $2\sigma^2$, and so $P_e(0) = Q(\frac{1}{\sqrt{2}\sigma})$. By symmetry $P_e(1) = Q(\frac{1}{\sqrt{2}\sigma})$, and so the error probability will be $P_e = \frac{1}{2}(P_e(0) + P_e(1)) = Q(\frac{1}{\sqrt{2}\sigma})$.

Now assume that we have access to Y_1, Y_2 and Y_3. Y_3 contains $Z_3 = Z_2$ under both hypotheses. Hence, $Y_2 - Y_3 = c_{i,2} + Z_2 - Z_3 = c_{i,2}$. This shows that at the receiver we can observe the second component of c_i without noise. As the second component is different under both hypotheses, we can make an error-free decision about H and the decision rule will be:

$$\hat{H} = \begin{cases} 0 & y_2 - y_3 = 0 \\ 1 & y_2 - y_3 = 1 \end{cases}$$

Clearly this decision rule minimizes the error probability. This shows once again that $(Y_1, Y_2)^\mathsf{T}$ can't be a sufficient statistic.

Solution 5. *(Signal translation)*

(a) Notice that

$$\|w_0(t)\|^2 = \|w_1(t)\|^2 = \int_0^{2T} w_0^2(t)dt = 2T$$

We first apply the Gram–Schmidt algorithm. We get the first basis vector from the first signal:

$$\psi_0(t) = \frac{w_0(t)}{\|w_0(t)\|} = \begin{cases} \frac{1}{\sqrt{2T}} & t \in [0, 2T] \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $\psi_0(t)$ and $w_1(t)$ are orthogonal. Thus we obtain the second basis vector by normalizing $w_1(t)$:

$$\psi_1(t) = \frac{w_1(t)}{\|w_1(t)\|} = \begin{cases} \frac{1}{\sqrt{2T}} & t \in [0, T] \\ -\frac{1}{\sqrt{2T}} & t \in [T, 2T] \\ 0 & \text{otherwise} \end{cases}$$

In the $\{\psi_0(t), \psi_1(t)\}$ basis, it is straightforward to see that $c_0 = (\sqrt{2T}, 0)^\mathsf{T}$ and $c_1 = (0, \sqrt{2T})^\mathsf{T}$.

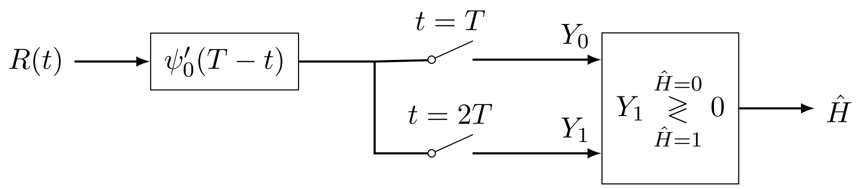
The other basis is the following:

$$\psi_0'(t) = \begin{cases} \frac{1}{\sqrt{T}} & t \in [0, T] \\ 0 & \text{otherwise} \end{cases} \qquad \psi_1'(t) = \begin{cases} \frac{1}{\sqrt{T}} & t \in [T, 2T] \\ 0 & \text{otherwise} \end{cases}$$

Observe that $\psi_1'(t) = \psi_0'(t - T)$. Hence, one matched filter at the receiver sampled twice suffices to project the received signal onto $\psi_0'(t)$ and $\psi_1'(t)$.

In the $\{\psi_0'(t), \psi_1'(t)\}$ basis, the codewords are $c_0 = (\sqrt{T}, \sqrt{T})^\mathsf{T}$ and $c_1 = (\sqrt{T}, -\sqrt{T})^\mathsf{T}$.

(b) The ML receiver is shown below.



Notice that Y_0 is not used. This is not surprising when we look at the signals: For $t \in [0, T]$, the two signals are identical.

(c) We calculate

$$\|w_0(t) - w_1(t)\| = 2\sqrt{T},$$

hence

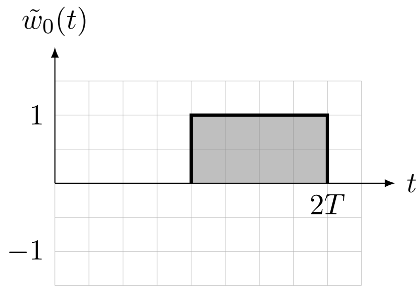
$$P_e = Q\left(\frac{\sqrt{T}}{\sqrt{N_0/2}}\right)$$

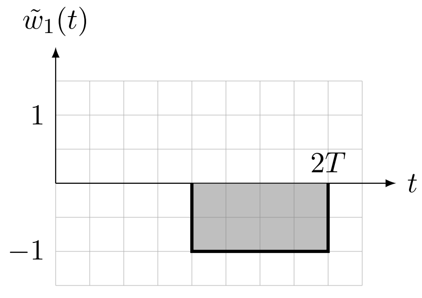
(d) Translating the signal points by any vector will not influence the error probability. However, if the translation vector is the center of mass of the original signal constellation, then the resulting signals will have minimum energy. We compute $v(t) = \frac{1}{2}w_0(t) + \frac{1}{2}w_1(t)$, thus

$$\tilde{w}_0(t) = w_0(t) - v(t) = \begin{cases} 1 & \text{for } t \in [T, 2T] \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{w}_1(t) = w_1(t) - v(t) = \begin{cases} -1 & \text{for } t \in [T, 2T] \\ 0 & \text{otherwise} \end{cases}$$

The resulting signal waveforms are shown below:





(e) The new signal constellation is antipodal. One can see that

$$\tilde{w}_0(t) = w_0(t) - v(t) = \frac{1}{2}w_0(t) - \frac{1}{2}w_1(t)$$

$$\tilde{w}_1(t) = w_1(t) - v(t) = \frac{1}{2}w_1(t) - \frac{1}{2}w_0(t) = -\tilde{w}_0(t)$$

This shows that we obtain an antipodal signal constellation regardless of the initial waveforms.

Solution 6. *(Orthogonal signal sets)*

(a) We first compute the centroid of the signal set:

$$a(t) = \sum_{j=0}^{m-1} P_H(j) w_j(t) = \frac{1}{m} \sum_{j=0}^{m-1} w_j(t)$$

The minimum-energy signal set is then obtained by translation:

$$\begin{aligned}\tilde{w}_j(t) &= w_j(t) - a(t) = w_j(t) - \frac{1}{m} \sum_{i=0}^{m-1} w_i(t) \\ &= \frac{m-1}{m} w_j(t) - \frac{1}{m} \sum_{i \neq j} w_i(t)\end{aligned}$$

(b)

$$\begin{aligned}\|\tilde{w}_j(t)\|^2 &= \langle \tilde{w}_j(t), \tilde{w}_j(t) \rangle \\ &= \langle \frac{m-1}{m} w_j(t) - \frac{1}{m} \sum_{i \neq j} w_j(t), \frac{m-1}{m} w_j(t) - \frac{1}{m} \sum_{k \neq j} w_k(t) \rangle \\ &= \left(\frac{m-1}{m} \right)^2 \mathcal{E} + \frac{1}{m^2} \sum_{i \neq j} \sum_{k \neq j} \langle w_i(t), w_k(t) \rangle \\ &= \left(\frac{m-1}{m} \right)^2 \mathcal{E} + \frac{m-1}{m^2} \mathcal{E} = \left(1 - \frac{1}{m} \right) \mathcal{E},\end{aligned}$$

and since all signals in $\tilde{\mathcal{W}}$ are equiprobable, we obtain $\tilde{\mathcal{E}} = \left(1 - \frac{1}{m}\right) \mathcal{E}$. The energy saving is therefore $\mathcal{E} - \tilde{\mathcal{E}} = \frac{1}{m} \mathcal{E}$. Alternatively, we could use $\mathcal{E} - \tilde{\mathcal{E}} = \|a(t)\|^2 = \frac{1}{m} \mathcal{E}$.

(c) Notice that $\sum_{j=0}^{m-1} \tilde{w}_j(t) = 0$ by the definition of $\tilde{w}_j(t)$, $j = 0, 1, \ldots, m-1$. Hence the m signals $\{\tilde{w}_0(t), \ldots, \tilde{w}_{m-1}(t)\}$ are linearly dependent. This means that their space has dimensionality less than m. We show that any collection of $m-1$ or less is linearly independent. That would prove that the dimensionality of the space $\{\tilde{w}_0(t), \ldots, \tilde{w}_{m-1}(t)\}$ is $m-1$. Without loss of essential generality we consider $\tilde{w}_0(t), \ldots, \tilde{w}_{m-2}(t)$. Assume that $\sum_{j=0}^{m-2} \alpha_j \tilde{w}_j(t) = 0$. Using the definition of $\tilde{w}_j(t)$, we may write

$$\sum_{j=0}^{m-2} \alpha_j \left(w_j(t) - \frac{1}{m} \sum_{i=0}^{m-1} w_i(t) \right) = 0,$$

$$\left(\sum_{j=0}^{m-2} \alpha_j w_j(t) \right) - \left(\frac{1}{m} \sum_{j=0}^{m-2} \alpha_j \right) \sum_{i=0}^{m-1} w_i(t) = 0,$$

$$\left(\sum_{j=0}^{m-2} \alpha_j w_j(t) \right) - \left(\beta \sum_{i=0}^{m-1} w_i(t) \right) = 0,$$

where $\beta = \frac{1}{m} \sum_{j=0}^{m-2} \alpha_j$. Therefore,

$$\sum_{j=0}^{m-2} (\alpha_j - \beta) w_j(t) - \beta w_{m-1}(t) = 0.$$

But $w_0(t), w_1(t), \ldots, w_{m-1}(t)$ is an orthogonal set and this implies $\beta = 0$ and $\alpha_j = \beta = 0,\ j = 0, 1, \ldots, m-2$. Hence $\tilde{w}_j(t),\ j = 0, 1, \ldots, m-2$ are linearly independent. We have proved that the new set spans a space of dimension $m-1$.