PROBLEM 1. Let $R$ and $\Phi$ be independent random variables. $R$ is distributed uniformly over the unit interval, $\Phi$ is distributed uniformly over the interval $[0, 2\pi)$.

(a) Interpret $R$ and $\Phi$ as the polar coordinates of a point in the plane. It is clear that the point lies inside (or on) the unit circle. Is the distribution of the point uniform over the unit disk? Take a guess!

(b) Define the random variables

$$X = R \cos \Phi$$
$$Y = R \sin \Phi$$

Find the joint distribution of the random variables $X$ and $Y$ by using the Jacobian determinant.

(c) Does the result of part (b) support or contradict your guess from part (a)? Explain.

PROBLEM 2. One of the two signals $c_0 = -1$, $c_1 = 1$ is transmitted over the channel shown in the left figure below. The two noise random variables $Z_1$ and $Z_2$ are statistically independent of the transmitted signal and of each other. Their density functions are

$$f_{Z_1}(\alpha) = f_{Z_2}(\alpha) = \frac{1}{2} e^{-|\alpha|}$$

(a) Derive a maximum likelihood decision rule.

(b) Describe the maximum likelihood decision regions in the $(y_1, y_2)$ plane. Describe also the “either choice” regions, i.e., the regions where it does not matter if you decide for $c_0$ or for $c_1$.

Hint: Use geometric reasoning and the fact that for a point $(y_1, y_2)$ as shown in the right figure above, $|y_1 - 1| + |y_2 - 1| = a + b$.

(c) A receiver decides that $c_1$ was transmitted if and only if $(y_1 + y_2) > 0$. Does this receiver minimize the error probability for equally likely messages?
(d) What is the error probability of the receiver in (c)?

*Hint:* One way to do this is to use the fact that if \( W = Z_1 + Z_2 \), then \( f_W(w) = \frac{e^{-w}}{1 + \omega} \) for \( \omega > 0 \) and \( f_W(-\omega) = f_W(\omega) \).

**Problem 3.** Use the Gram–Schmidt procedure to find an orthonormal basis for the vector space spanned by the functions \( \{w_0(t), w_1(t)\} \) below.

\[
\begin{align*}
\quad & w_0(t) \\
\quad & w_1(t)
\end{align*}
\]

**Problem 4.**

(a) By means of the Gram–Schmidt procedure, find an orthonormal basis for the space spanned by the waveforms \( \{\beta_0(t), \beta_1(t), \beta_2(t)\} \) below.

\[
\begin{align*}
\quad & \beta_0(t) \\
\quad & \beta_1(t) \\
\quad & \beta_2(t)
\end{align*}
\]

(b) In your chosen orthonormal basis, let \( w_0(t) \) and \( w_1(t) \) be represented by the codewords \( c_0 = (3, -1, 1)^T \) and \( c_1 = (-1, 2, 3)^T \) respectively. Plot \( w_0(t) \) and \( w_1(t) \).

(c) Compute the (standard) inner products \( \langle c_0, c_1 \rangle \) and \( \langle w_0, w_1 \rangle \) and compare them.

(d) Compute the norms \( ||c_0|| \) and \( ||w_0|| \) and compare them.

**Problem 5.** Let \( N(t) \) be white Gaussian noise of power spectral density \( \frac{N_0}{2} \). Let \( g_1(t) \), \( g_2(t) \), and \( g_3(t) \) be waveforms as shown below. For \( i = 1, 2, 3 \), let \( Z_i = \int N(t)g_i^*(t)dt \), \( Z = (Z_1, Z_2)^T \), and \( U = (Z_1, Z_3)^T \).

\[
\begin{align*}
\quad & g_1(t) \\
\quad & g_2(t) \\
\quad & g_3(t)
\end{align*}
\]
(a) Determine the norm \( \|g_i\|, i = 1, 2, 3 \).

(b) Are \( Z_1 \) and \( Z_2 \) independent? Justify your answer.

Consider now the regions depicted below:

(c) Find the probability \( P_a \) that \( Z \) lies in the square of the left figure.

(d) Find the probability \( P_b \) that \( Z \) lies in the square of the middle figure.

(e) Find the probability \( Q_a \) that \( U \) lies in the square of the left figure.

(f) Find the probability \( Q_b \) that \( U \) lies in the square of the right figure.

**Problem 6.** Consider the four sinusoid waveforms \( w_k(t), k = 0, 1, 2, 3 \) represented in the figure below.

(a) Determine an orthonormal basis for the signal space spanned by these waveforms.

*Hint:* No lengthy calculations needed.

(b) Determine the codewords \( c_i, i = 0, 1, 2, 3 \) representing the waveforms.

(c) Assume a transmitter sends \( w_i \) to communicate a digit \( i \in \{0, 1, 2, 3\} \) across a continuous-time AWGN channel of power spectral density \( \frac{N_0}{2} \). Write an expression for the error probability of the ML receiver in terms of \( \mathcal{E} \) and \( N_0 \).