ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

**Handout 11** Principles of Digital Communications
Problem Set 5 Mar. 22, 2017

PROBLEM 1. Let $R$ and $\Phi$ be independent random variables. $R$ is distributed uniformly over the unit interval, $\Phi$ is distributed uniformly over the interval $[0, 2\pi)$.

(a) Interpret $R$ and $\Phi$ as the polar coordinates of a point in the plane. It is clear that the point lies inside (or on) the unit circle. Is the distribution of the point uniform over the unit disk? Take a guess!

(b) Define the random variables

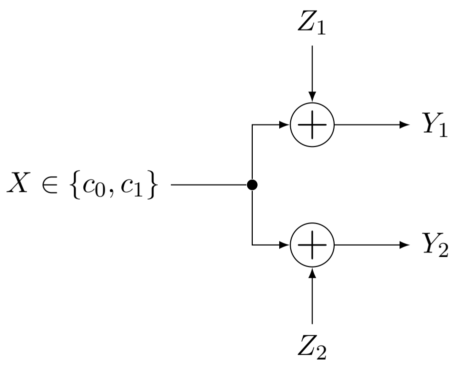
$$X = R\cos\Phi$$
$$Y = R\sin\Phi$$

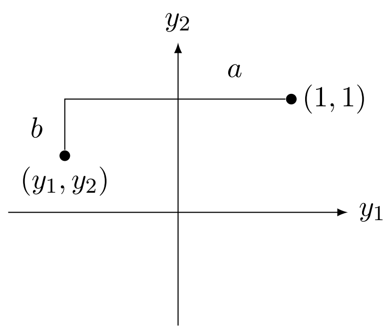
Find the joint distribution of the random variables $X$ and $Y$ by using the Jacobian determinant.

(c) Does the result of part (b) support or contradict your guess from part (a)? Explain.

PROBLEM 2. One of the two signals $c_0 = -1$, $c_1 = 1$ is transmitted over the channel shown in the left figure below. The two noise random variables $Z_1$ and $Z_2$ are statistically independent of the transmitted signal and of each other. Their density functions are

$$f_{Z_1}(\alpha) = f_{Z_2}(\alpha) = \frac{1}{2}e^{-|\alpha|}$$




(a) Derive a maximum likelihood decision rule.

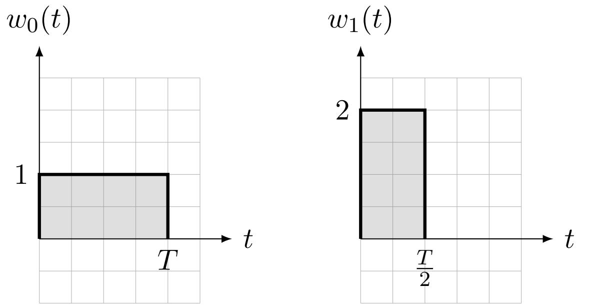
(b) Describe the maximum likelihood decision regions in the $(y_1, y_2)$ plane. Describe also the "either choice" regions, i.e., the regions where it does not matter if you decide for $c_0$ or for $c_1$.
*Hint:* Use geometric reasoning and the fact that for a point $(y_1, y_2)$ as shown in the right figure above, $|y_1 - 1| + |y_2 - 1| = a + b$.

(c) A receiver decides that $c_1$ was transmitted if and only if $(y_1 + y_2) > 0$. Does this receiver minimize the error probability for equally likely messages?

(d) What is the error probability of the receiver in (c)?

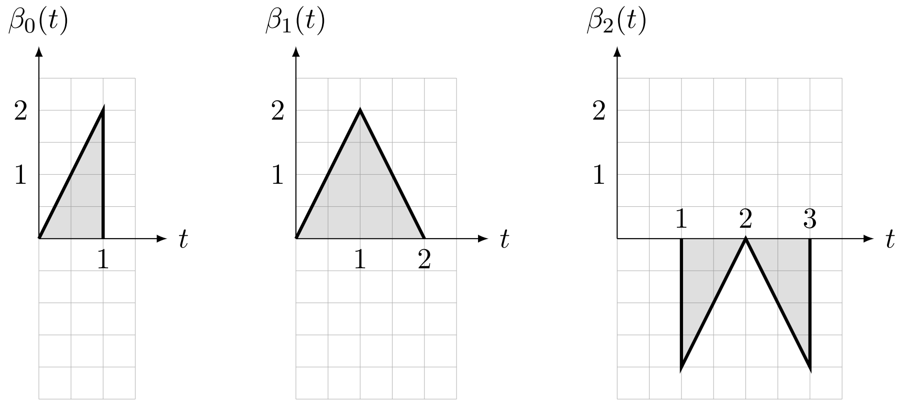
*Hint:* One way to do this is to use the fact that if $W = Z_1 + Z_2$, then $f_W(w) = \frac{e^{-\omega}}{4}(1+\omega)$ for $\omega > 0$ and $f_W(-\omega) = f_W(\omega)$.

Problem 3. Use the Gram–Schmidt procedure to find an orthonormal basis for the vector space spanned by the functions $\{w_0(t), w_1(t)\}$ below.




Problem 4.

(a) By means of the Gram–Schmidt procedure, find an orthonormal basis for the space spanned by the waveforms $\{\beta_0(t), \beta_1(t), \beta_2(t)\}$ below.

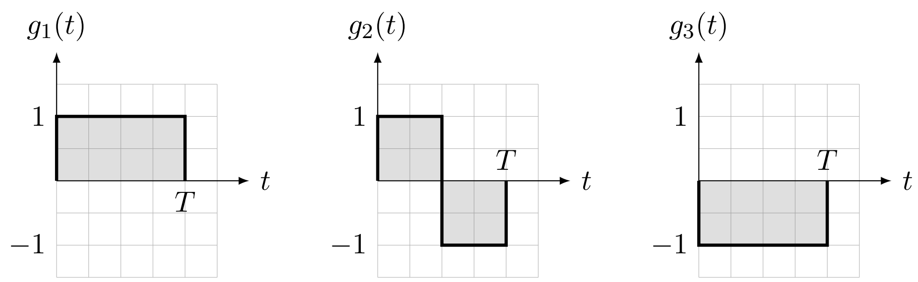



(b) In your chosen orthonormal basis, let $w_0(t)$ and $w_1(t)$ be represented by the codewords $c_0 = (3, -1, 1)^\mathsf{T}$ and $c_1 = (-1, 2, 3)^\mathsf{T}$ respectively. Plot $w_0(t)$ and $w_1(t)$.

(c) Compute the (standard) inner products $\langle c_0, c_1 \rangle$ and $\langle w_0, w_1 \rangle$ and compare them.

(d) Compute the norms $\|c_0\|$ and $\|w_0\|$ and compare them.

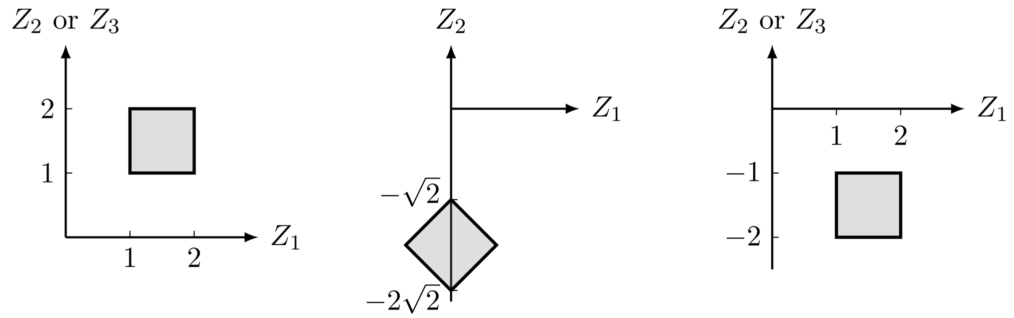
Problem 5. Let $N(t)$ be white Gaussian noise of power spectral density $\frac{N_0}{2}$. Let $g_1(t)$, $g_2(t)$, and $g_3(t)$ be waveforms as shown below. For $i = 1, 2, 3$, let $Z_i = \int N(t) g_i^*(t) dt$, $Z = (Z_1, Z_2)^\mathsf{T}$, and $U = (Z_1, Z_3)^\mathsf{T}$.



(a) Determine the norm $\|g_i\|$, $i = 1, 2, 3$.

(b) Are $Z_1$ and $Z_2$ independent? Justify your answer.

Consider now the regions depicted below:



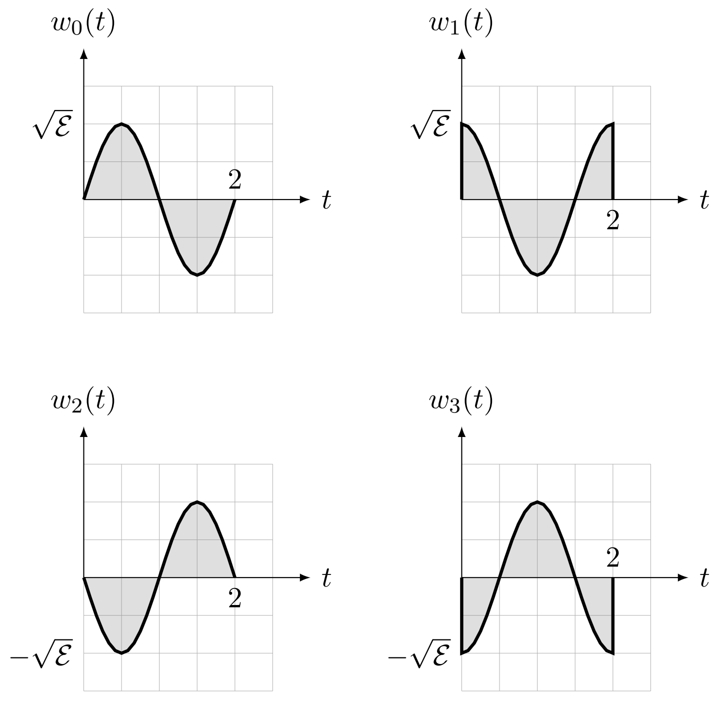

(c) Find the probability $P_a$ that $Z$ lies in the square of the left figure.

(d) Find the probability $P_b$ that $Z$ lies in the square of the middle figure.

(e) Find the probability $Q_a$ that $U$ lies in the square of the left figure.

(f) Find the probability $Q_b$ that $U$ lies in the square of the right figure.

PROBLEM 6. Consider the four sinusoid waveforms $w_k(t)$, $k = 0, 1, 2, 3$ represented in the figure below.




(a) Determine an orthonormal basis for the signal space spanned by these waveforms.
*Hint:* No lengthy calculations needed.

(b) Determine the codewords $c_i$, $i = 0, 1, 2, 3$ representing the waveforms.

(c) Assume a transmitter sends $w_i$ to communicate a digit $i \in \{0, 1, 2, 3\}$ across a continuous-time AWGN channel of power spectral density $\frac{N_0}{2}$. Write an expression for the error probability of the ML receiver in terms of $\mathcal{E}$ and $N_0$.