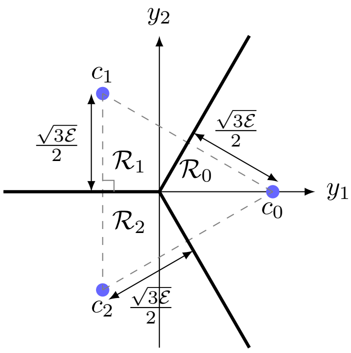
Solution 1.

(a) Since the hypotheses are equally likely, the optimal decision rule is the ML rule and since the noise is additive, white and Gaussian (AWGN), this reduces to the minimum distance decision rule:



(b) For the hypotheses with unequal prior, one has to use the MAP decision rule in order to minimize the error probability. The MAP decision rule decides $\hat{H} = i$ iff

$$f_{Y|H}(y|i)P_H(i) > f_{Y|H}(y|j)P_H(j) \qquad \forall j \neq i,$$

or equivalently (by dividing both sides of the above by $f_{Y|H}(y|0)$),

$$\frac{f_{Y|H}(y|i)}{f_{Y|H}(y|0)}P_H(i) > \frac{f_{Y|H}(y|j)}{f_{Y|H}(y|0)}P_H(j) \qquad \forall j \neq i,$$

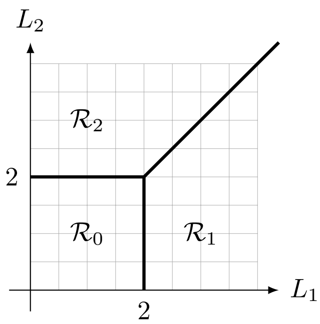
which means

$$\frac{L_i}{L_j} > \frac{P_H(j)}{P_H(i)} \qquad \forall j \neq i,$$

(note that $L_0 = \frac{f_{Y|H}(Y|0)}{f_{Y|H}(Y|0)} = 1$). Thus, since $P_H(0) = 2P_H(1) = 2P_H(2)$,

$$\mathcal{R}_0 = \{(L_1, L_2) : L_1 < 2, L_2 < 2\},$$
$$\mathcal{R}_1 = \{(L_1, L_2) : L_1 > 2, L_1 > L_2\},$$
$$\mathcal{R}_2 = \{(L_1, L_2) : L_2 > 2, L_2 > L_1\}.$$

The regions are plotted in the figure below:



Remark. *You must have noticed that the solution is independent of the noise distribution and the hypotheses setup. The MAP decision regions for any ternary hypothesis testing problem with priors $P_H(0) = 2P_H(1) = 2P_H(2)$ will be what we found here.*

Solution 2.

(a) One can see that the event $\{X \in \text{Region}\}$ only depends on the first component X_1. Hence, we have

$$\begin{aligned}\Pr\{X \in \text{Region}\} &= \Pr\{(X_1 \geq -2) \cap (X_1 \leq 1)\} \\ &= 1 - \Pr\{(X_1 < -2) \cup (X_1 > 1)\} \\ &= 1 - Q\left(\frac{2}{\sigma}\right) - Q\left(\frac{1}{\sigma}\right),\end{aligned}$$

where the last equality is true because $\{X_1 < -2\}$ and $\{X_1 > 1\}$ are disjoint events.

(b) Because X_1 and X_2 are independent and have the same variance, rotating the vector X by any angle around the origin does not change its distribution. Equivalently, we can rotate the square region in Figure (b) by 45 degrees, and the probability of X being in the rotated region is the same as for the original region. The new region is a square whose edges are parallel to the axes of the coordinate system. The points where the edges of the square intersect the axes are $(\sqrt{2}, 0)$, $(-\sqrt{2}, 0)$, $(0, \sqrt{2})$ and $(0, -\sqrt{2})$. Hence,

$$\begin{aligned}\Pr\{X \in \text{Region}\} &= \Pr\{(-\sqrt{2} \leq X_1 \leq \sqrt{2}) \cap (-\sqrt{2} \leq X_2 \leq \sqrt{2})\} \\ &\overset{(1)}{=} \Pr\{-\sqrt{2} \leq X_1 \leq \sqrt{2}\}^2 \\ &= \left[1 - \Pr\{(X_1 < -\sqrt{2}) \cup (X_1 > \sqrt{2})\}\right]^2 \\ &= \left[1 - 2Q\left(\frac{\sqrt{2}}{\sigma}\right)\right]^2,\end{aligned}$$

where (1) holds because X_1 and X_2 are independent and identically distributed.

(c) We solve this part in three different ways:

i. As in the previous part, we can rotate X such that one of its components, say X_1, is perpendicular to the straight line that delimits the shaded region. Then, we need to know the shortest distance d of that line to the origin (the length of a segment that starts at $(0, 0)$ and is perpendicular to the line). Using standard trigonometric techniques, one finds that this length is $d = \frac{2}{\sqrt{5}}$. Then, it follows that

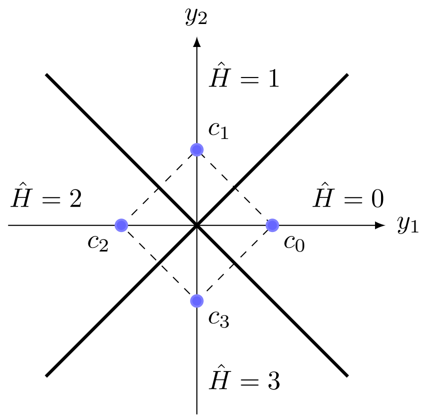
$$\begin{aligned}\Pr\{X \in \text{Region}\} &= \Pr\{X_1 \geq \frac{2}{\sqrt{5}}\} \\ &= Q\left(\frac{2}{\sqrt{5}\sigma}\right).\end{aligned}$$

ii. We are looking for the probability that $X_2 \geq 1 - \frac{1}{2}X_1$, i.e., the probability that $Z \triangleq X_2 + \frac{1}{2}X_1 - 1 \geq 0$. But $Z \sim \mathcal{N}(-1, \frac{5}{4}\sigma^2)$. Hence, $\Pr\{X \in \text{Region}\} = \Pr\{Z \geq 0\} = Q\left(\frac{2}{\sqrt{5}\sigma}\right)$.

iii. We project $X = (X_1, X_2)^\intercal$ to the vector perpendicular to the line that delimits the shaded region. The length of the projection is $Z \sim \mathcal{N}(0, \sigma^2)$. The sought probability is $\Pr\{Z \geq d\} = Q\left(\frac{d}{\sigma}\right) = Q\left(\frac{2}{\sqrt{5}\sigma}\right)$, where d is the distance from the delimiting line to the origin.

Solution 3.

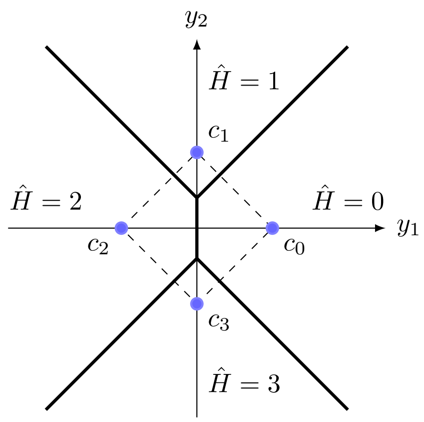
(a) If $P_H(i)$ is the same for all i, then the decision regions are as follows:



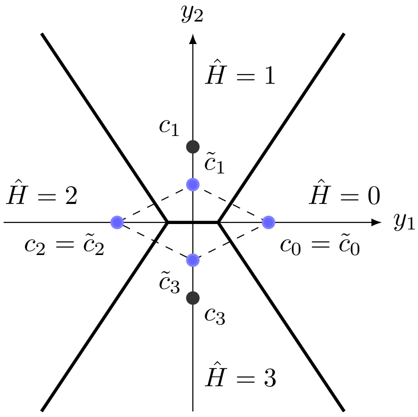
(b) The decision boundary between two hypotheses $\hat{H} = i$ and $\hat{H} = j$ is given by

$$\|Y - c_i\|^2 - \|Y - c_j\|^2 = 2\sigma^2 \ln \frac{P_H(i)}{P_H(j)}.$$

This is an affine plane perpendicular to the segment that joins c_i to c_j. If $P_H(i) > P_H(j)$, then the affine plane is shifted away from c_i, to increase $\mathcal{R}_i$. The decision regions for this case are given below:



(c) Define a new observation $\tilde{Y} = (Y_1, Y_2/2)$. The new observation $\tilde{Y}$ is a sufficient statistic because we can determine Y from $\tilde{Y}$. Thus the receiver observes $\tilde{Y} = \tilde{c}_i + \tilde{Z}$, where $\tilde{c}_i = (c_{i1}, c_{i2}/2)$ and $\tilde{Z} = (Z_1, Z_2/2)$. Note that in this new setup we have $\tilde{c}_0 = c_0$, $\tilde{c}_1 = c_1/2$, $\tilde{c}_2 = c_2$, $\tilde{c}_3 = c_3/2$ and $\tilde{Z} \sim \mathcal{N}(0, \sigma^2 I_2)$. The decision regions for this case are given below:



SOLUTION 4. Since Z_1 and Z_2 don't have the same variance, the noise is not white, and so we cannot directly apply the results for discrete time AWGN channels which we are familiar with. A smart way to solve this problem is to apply a transformation on $Y = (Y_1, Y_2)^\mathsf{T}$ to get a sufficient statistic $\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2)^\mathsf{T}$ that can be seen as the output of a discrete time AWGN channel.

Since Z_1 and Z_2 are independent and have variance σ_1^2 and σ_2^2 respectively, $\frac{Z_1}{\sigma_1}$ and $\frac{Z_2}{\sigma_2}$ are independent and have variance 1. Thus, $(\frac{Z_1}{\sigma_1}, \frac{Z_2}{\sigma_2})^\mathsf{T} \sim \mathcal{N}(0, I_2)$ which is a white noise of power 1. Therefore, if we define

$$\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2)^\mathsf{T} \triangleq \left(\frac{Y_1}{\sigma_1}, \frac{Y_2}{\sigma_2}\right)^\mathsf{T},$$

and

$$\tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2)^\mathsf{T} \triangleq \left(\frac{Z_1}{\sigma_1}, \frac{Z_2}{\sigma_2}\right)^\mathsf{T},$$

we will have

$$\begin{aligned} \tilde{Y} &= \tilde{c}_0 + \tilde{Z} && \text{if } H = 0 \text{ and} \\ \tilde{Y} &= \tilde{c}_1 + \tilde{Z} && \text{if } H = 1, \end{aligned}$$

where $\tilde{c}_0 = (\frac{A}{\sigma_1}, \frac{A}{\sigma_2})^\mathsf{T}$, $\tilde{c}_1 = (-\frac{A}{\sigma_1}, -\frac{A}{\sigma_2})^\mathsf{T}$ and $\tilde{Z} \sim \mathcal{N}(0, I_2)$. It is clear that $\tilde{Y}$ can be seen as the output of a discrete time AWGN channel (with two observations), which is a situation we are familiar with and know very well how to handle.

Another solution for the problem is to start from the basic principles, i.e., computing the probability densities $f_{Y|H}$ and probabilities $P_{H|Y}$, then computing the decision regions and error probabilities without relying on the results of discrete time AWGN channels.

We provide the two solutions here. While the second solution starts from the basic principles, the first one builds on results and intuitions that we have already developed.

FIRST SOLUTION:

(a) Since $\tilde{Z} \sim \mathcal{N}(0, I_2)$, the line that separates the two decision regions in the $\tilde{y}$-plane is the perpendicular bisector of the segment $[\tilde{c}_0 \ \tilde{c}_1]$ (i.e., the line that has $\tilde{c}_0 - \tilde{c}_1$ as a normal vector and passes through the midpoint of $\tilde{c}_0$ and $\tilde{c}_1$ — which is the origin).

Therefore, the MAP decision regions in the $\tilde{y}$-plane are given by

$$\langle \tilde{y}, \tilde{c}_0 - \tilde{c}_1 \rangle \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0, \text{ or equivalently,}$$

$$\tilde{y}_1 \frac{2A}{\sigma_1} + \tilde{y}_2 \frac{2A}{\sigma_2} \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0,$$

$$\frac{\tilde{y}_1}{\sigma_1} + \frac{\tilde{y}_2}{\sigma_2} \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0.$$

Now since $\tilde{y}_1 = \frac{y_1}{\sigma_1}$ and $\tilde{y}_2 = \frac{y_2}{\sigma_2}$, the MAP decision regions in the y-plane are given by

$$\frac{y_1}{\sigma_1^2} + \frac{y_2}{\sigma_2^2} \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0, \text{ or equivalently,}$$

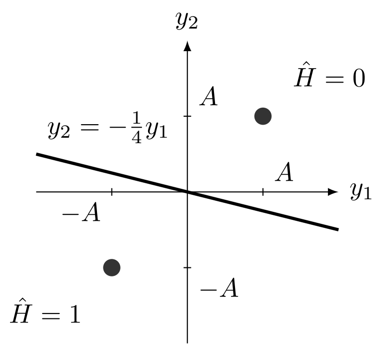
$$\sigma_2^2 y_1 + \sigma_1^2 y_2 \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0.$$

(b) When $\sigma_1 = 2\sigma_2$, the decision rule becomes

$$\sigma_2^2 y_1 + 4\sigma_2^2 y_2 \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0, \text{ or equivalently,}$$

$$y_2 \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} -\frac{y_1}{4}.$$

The decision regions are sketched below:



(c) We compute the probability of error based on $\tilde{Y}$ and $\tilde{Z}$. The distance between $\tilde{c}_0$ and the separator line is equal to

$$\|\tilde{c}_0\| = A\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}.$$

Since $\tilde{Z} \sim \mathcal{N}(0, I_2)$, we have

$$P_e(0) = Q\left(A\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\right).$$

Similarly, we have

$$P_e(1) = Q\left(A\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\right).$$

Therefore,

$$P_e = P_e(0)P_H(0) + P_e(1)P_H(1) = Q\left(A\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\right).$$

SECOND SOLUTION:

(a) We have

$$\begin{aligned} f_{Y|H}(y|0) &= \frac{1}{2\pi\sigma_1\sigma_2}\exp\left[-\frac{(y_1-A)^2}{2\sigma_1^2} - \frac{(y_2-A)^2}{2\sigma_2^2}\right] \\ f_{Y|H}(y|1) &= \frac{1}{2\pi\sigma_1\sigma_2}\exp\left[-\frac{(y_1+A)^2}{2\sigma_1^2} - \frac{(y_2+A)^2}{2\sigma_2^2}\right]. \end{aligned}$$

The MAP decision rule is

$$\frac{f_{Y|H}(y|1)}{f_{Y|H}(y|0)} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{P_H(0)}{P_H(1)}$$

or, by taking the logarithm,

$$\begin{aligned} \ln\left[\frac{f_{Y|H}(y|1)}{f_{Y|H}(y|0)}\right] &\underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \ln\left[\frac{P_H(0)}{P_H(1)}\right], \text{ or equivalently,} \\ \frac{2Ay_1}{\sigma_1^2} + \frac{2Ay_2}{\sigma_2^2} &\underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0, \\ \sigma_2^2 y_1 + \sigma_1^2 y_2 &\underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0. \end{aligned}$$

(b) Refer to the first solution.

(c) We first determine the probability of error when $H = 1$:

$$P_e(1) = \Pr\{\sigma_2^2 Y_1 + \sigma_1^2 Y_2 > 0 | H = 1\}.$$

If $H = 1$, $\sigma_2^2 Y_1 + \sigma_1^2 Y_2 = \sigma_2^2(-A + Z_1) + \sigma_1^2(-A + Z_2)$. We see immediately that this is normally distributed, $\sim \mathcal{N}(-A(\sigma_2^2 + \sigma_1^2), (\sigma_2^4\sigma_1^2 + \sigma_1^4\sigma_2^2))$. Hence,

$$\begin{aligned} P_e(1) &= Q\left(\frac{A(\sigma_2^2+\sigma_1^2)}{\sqrt{\sigma_2^4\sigma_1^2 + \sigma_1^4\sigma_2^2}}\right) \\ &= Q\left(A\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\right). \end{aligned}$$

Similarly,

$$P_e(0) = Q\left(A\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\right),$$

and

$$P_e = P_e(0)P_H(0) + P_e(1)P_H(1) = Q\left(A\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\right).$$

Solution 5.

(a) Our observation is $Y = AX + Z$. The conditional p.d.f. of Y under the hypothesis $H = 0$ can be computed in the following manner:

$$\begin{aligned} f_{Y|H}(y|0) &= f_{Y|H,A}(y|0,0)P_A(0) + f_{Y|H,A}(y|0,1)P_A(1) \\ &= \frac{1}{2}f_Z(y) + \frac{1}{2}f_Z(y+b) \\ &= \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\left(e^{-\frac{y^2}{2\sigma^2}} + e^{-\frac{(y+b)^2}{2\sigma^2}}\right). \end{aligned}$$

In the same way, we have

$$f_{Y|H}(y|1) = \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\left(e^{-\frac{y^2}{2\sigma^2}} + e^{-\frac{(y-b)^2}{2\sigma^2}}\right).$$

Writing the ML decision rule in this case, we get

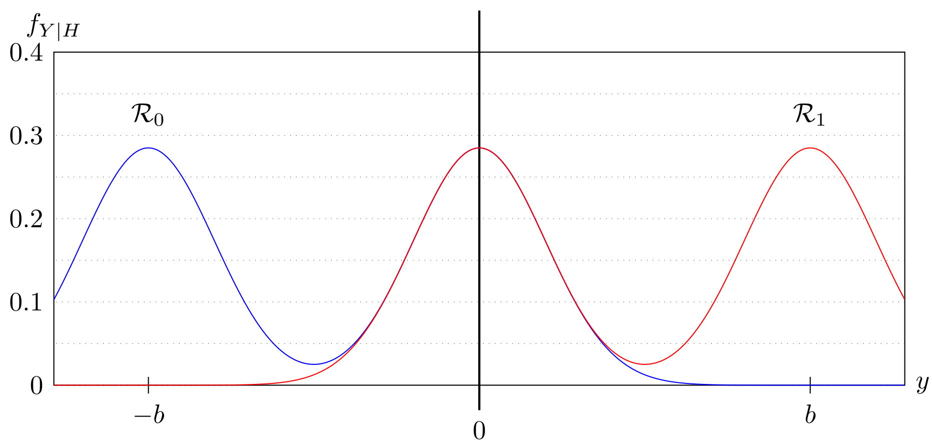
$$\frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\left(e^{-\frac{y^2}{2\sigma^2}} + e^{-\frac{(y+b)^2}{2\sigma^2}}\right) \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\left(e^{-\frac{y^2}{2\sigma^2}} + e^{-\frac{(y-b)^2}{2\sigma^2}}\right),$$

which is equivalent to

$$e^{-\frac{(y+b)^2}{2\sigma^2}} \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} e^{-\frac{(y-b)^2}{2\sigma^2}}, \text{ or, after taking the logarithm,}$$

$$0 \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} y.$$

Thus, we get a familiar problem and we see immediately that our ML rule decides for $H = 0$ when $y \leq 0$ and for $H = 1$ when $y > 0$. The decision regions are shown in the figure below:



(b) By symmetry, we have

$$\begin{aligned} P_e &= P_e(0) = P_e(1) \\ &= \Pr\{y > 0 | H = 0\} \\ &= \int_0^\infty f_{Y|H}(y|0)\, dy \\ &= \int_0^\infty \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\left(e^{-\frac{y^2}{2\sigma^2}} + e^{-\frac{(y+b)^2}{2\sigma^2}}\right) dy \\ &= \frac{1}{2}Q(0) + \frac{1}{2}Q\left(\frac{b}{\sigma}\right) \\ &= \frac{1}{4} + \frac{1}{2}Q\left(\frac{b}{\sigma}\right). \end{aligned}$$

Solution 6.

(a) *16-PAM.* Denote the additive white Gaussian noise process by Z. Thus, Z is zero-mean Gaussian of variance σ^2, and the observation Y is also Gaussian of variance σ^2, but with mean corresponding to the particular signal point that is being transmitted. If H is the hypothesis and we label the signal points from left to right by $1, \ldots, 16$, then

$$\begin{aligned} P_e(1) &= \Pr\{Y \geq -7a | H = 1\} = \Pr\{Z \geq \frac{a}{2}\} \\ &= \Pr\{\frac{Z}{\sigma} \geq \frac{a}{2\sigma}\} = Q\left(\frac{a}{2\sigma}\right). \end{aligned}$$

By symmetry, $P_e(1) = P_e(16)$.

Moreover,

$$\begin{aligned} P_e(2) &= \Pr\{(Y \leq -7a) \cup (Y \geq -6a) | H = 2\} \\ &= \Pr\{\left(Z \leq -\frac{a}{2}\right) \cup \left(Z \geq \frac{a}{2}\right)\} = 2\Pr\{Z \geq \frac{a}{2}\} \\ &= 2Q\left(\frac{a}{2\sigma}\right). \end{aligned}$$

Again, by symmetry, $P_e(i) = P_e(2)$, for $i = 3, \ldots, 15$. Putting things together, we obtain

$$\begin{aligned} P_e &= \sum_{i=1}^{16} P_H(i) P_e(i) = \sum_{i=1}^{16} \frac{1}{16} P_e(i) \\ &= \frac{1}{16}\left(2 \cdot Q\left(\frac{a}{2\sigma}\right) + 14 \cdot 2Q\left(\frac{a}{2\sigma}\right)\right) \\ &= \frac{15}{8} Q\left(\frac{a}{2\sigma}\right). \end{aligned}$$

16-QAM. Denote the additive white Gaussian noise process in the x_1-direction by Z_1 and in the x_2-direction by Z_2. In our setup, both Z_1 and Z_2 are zero-mean Gaussian of variance σ^2. Label the signal points from left to right, top to bottom by $1, \ldots, 16$. Then, for the four corner points, we find

$$P_e(1) = \Pr\{(Y_1 \geq -b) \cup (Y_2 \leq b) | H = 1\}.$$

Notice that $\{Y_1 \geq -b\}$ and $\{Y_2 \leq b\}$ are not disjoint events, so

$$P_e(1) = \Pr\{Y_1 \geq -b | H=1\} + \Pr\{Y_2 \leq b | H=1\} - \Pr\{(Y_1 \geq -b) \cap (Y_2 \leq b) | H=1\}.$$

An alternative (and somewhat simpler) approach is to compute the probability of the correct decision, $P_c(1)$, and then determine $P_e(1) = 1 - P_c(1)$. Thus,

$$\begin{aligned}
P_c(1) &= \Pr\{(Y_1 \leq -b) \cap (Y_2 \geq b) | H=1\} \\
&= \Pr\{(Y_1 \leq -b | H=1\} \Pr\{Y_2 \geq b | H=1\} \\
&= \Pr\{Z_1 \leq \frac{b}{2}\} \Pr\{Z_2 \geq -\frac{b}{2}\} \\
&= \left(1 - Q\left(\frac{b}{2\sigma}\right)\right) Q\left(-\frac{b}{2\sigma}\right) \\
&= \left(1 - Q\left(\frac{b}{2\sigma}\right)\right)^2.
\end{aligned}$$

For the points on the edges (i.e., numbers $2, 3, 5, 8, 9, 12, 14, 15$), we find similarly

$$\begin{aligned}
P_c(2) &= \Pr\{(-b \leq Y_1 \leq 0) \cap (Y_2 \geq b) | H=2\} \\
&= \Pr\{-\frac{b}{2} \leq Z_1 \leq \frac{b}{2}\} \Pr\{Z_2 \geq -\frac{b}{2}\},
\end{aligned}$$

where

$$\begin{aligned}
\Pr\{-\frac{b}{2} \leq Z_1 \leq \frac{b}{2}\} &= 1 - \Pr\{\left(Z_1 \leq -\frac{b}{2}\right) \cup \left(Z_1 \geq \frac{b}{2}\right)\} \\
&= 1 - 2\Pr\{Z_1 \geq \frac{b}{2}\} \\
&= 1 - 2Q\left(\frac{b}{2\sigma}\right),
\end{aligned}$$

thus,

$$P_c(2) = \left(1 - 2Q\left(\frac{b}{2\sigma}\right)\right)\left(1 - Q\left(\frac{b}{2\sigma}\right)\right).$$

Finally, for the four points in the middle, we obtain

$$\begin{aligned}
P_c(6) &= \Pr\{(-b \leq Y_1 \leq 0) \cap (0 \leq Y_2 \leq b) | H=6\} \\
&= \Pr\{-\frac{b}{2} \leq Z_1 \leq \frac{b}{2}\} \Pr\{-\frac{b}{2} \leq Z_2 \leq \frac{b}{2}\} \\
&= \left(1 - 2Q\left(\frac{b}{2\sigma}\right)\right)^2.
\end{aligned}$$

Putting things together, we find

$$\begin{aligned}
P_c &= \sum_{i=1}^{16} P_H(i) P_c(i) = \sum_{i=1}^{16} \frac{1}{16} P_c(i) \\
&= \frac{1}{16}\Bigg[4\left(1 - Q\left(\frac{b}{2\sigma}\right)\right)^2 + 8\left(1 - Q\left(\frac{b}{2\sigma}\right)\right)\left(1 - 2Q\left(\frac{b}{2\sigma}\right)\right) \\
&\qquad + 4\left(1 - 2Q\left(\frac{b}{2\sigma}\right)\right)^2\Bigg] \\
&= 1 - 3Q\left(\frac{b}{2\sigma}\right) + \frac{9}{4}\left(Q\left(\frac{b}{2\sigma}\right)\right)^2.
\end{aligned}$$

From here, we find $P_e = 1 - P_c$, thus

$$P_e = 3Q\left(\frac{b}{2\sigma}\right) - \frac{9}{4}\left(Q\left(\frac{b}{2\sigma}\right)\right)^2.$$

(b) *16-PAM.* By symmetry, we only consider the positive signals to find

$$\begin{aligned}\mathcal{E} &= 2\frac{1}{16}\left(\left(\frac{a}{2}\right)^2 + \left(\frac{3a}{2}\right)^2 + \ldots + \left(\frac{15a}{2}\right)^2\right) \\ &= \frac{a^2}{32}\left(1 + 3^2 + 5^2 + \ldots + 15^2\right) = \frac{85a^2}{4}.\end{aligned}$$

16-QAM. By symmetry, we only consider the first quadrant to find

$$\begin{aligned}\mathcal{E} &= 4\frac{1}{16}\left(\left[\left(\frac{b}{2}\right)^2 + \left(\frac{b}{2}\right)^2\right] + \left[\left(\frac{3b}{2}\right)^2 + \left(\frac{3b}{2}\right)^2\right] + 2\left[\left(\frac{b}{2}\right)^2 + \left(\frac{3b}{2}\right)^2\right]\right) \\ &= \frac{b^2}{16}\left(1 + 1 + 9 + 9 + 2(1 + 9)\right) = \frac{5b^2}{2}.\end{aligned}$$

(c) *16-PAM.* We find $a/2 = \sqrt{\mathcal{E}/85}$, thus

$$P_e = \frac{15}{8}Q\left(\sqrt{\frac{\mathcal{E}}{85\sigma^2}}\right).$$

16-QAM. We find $b/2 = \sqrt{\mathcal{E}/10}$, thus

$$P_e = 3Q\left(\sqrt{\frac{\mathcal{E}}{10\sigma^2}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{\mathcal{E}}{10\sigma^2}}\right).$$