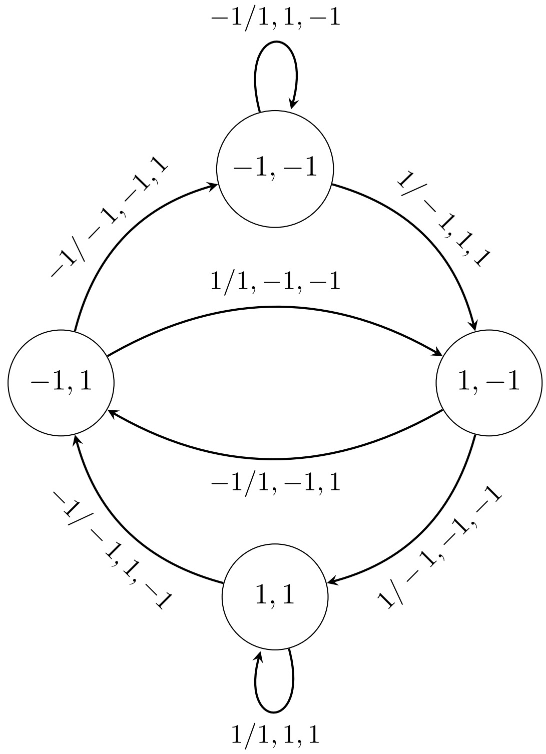
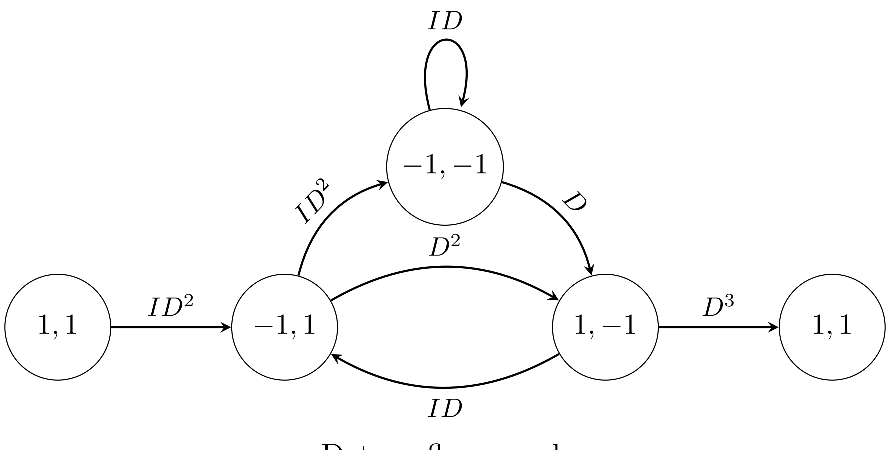
Solution 1.

(a) The state diagram and detour flow graph are respectively shown below:



State diagram



Detour flow graph

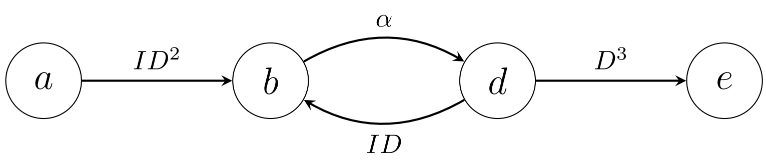
(b) Let a, b, c, d, e respectively represent the states $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$ and $(1, 1)$. We have

$$\begin{aligned} T_b &= T_d ID + T_a ID^2 \\ T_c &= T_c ID + T_b ID^2 \\ T_d &= T_b D^2 + T_c D. \end{aligned}$$

Substituting $T_c = T_b \frac{ID^2}{1-ID}$ in the third equation above,

$$\begin{aligned} T_d &= T_b D^2 + T_b \frac{ID^3}{1-ID} \\ &= T_b \left(D^2 + \frac{ID^3}{1-ID} \right) \\ &= T_b \frac{D^2}{1-ID} \\ &= T_b \alpha, \end{aligned}$$

with $\alpha = \frac{D^2}{1-ID}$. The detour flow graph can thus be simplified to:



In $T_b = T_d ID + T_a ID^2$, we substitute for T_d to get

$$T_b = T_a \frac{ID^2(1-ID)}{1-ID-ID^3}.$$

It follows that

$$T_d = T_b \frac{D^2}{1-ID} = T_a \frac{ID^4}{1-ID-ID^3},$$

and that

$$T(I,D) = T_e = T_a \frac{ID^7}{1-ID-ID^3}.$$

Taking the derivative yields

$$\frac{\partial T(I,D)}{\partial I} = \frac{D^7(1-ID-ID^3) - ID^7(-D-D^3)}{(1-ID-ID^3)^2} = \frac{D^7}{(1-ID-ID^3)^2}.$$

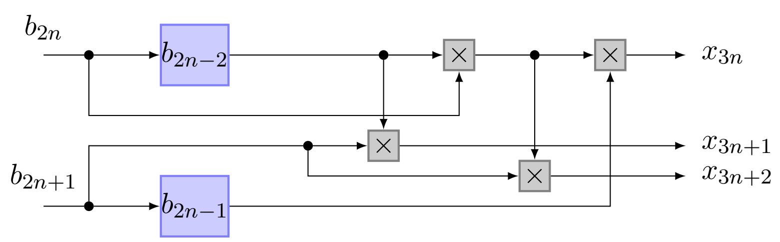
Therefore, we find

$$\begin{aligned} P_b &\leq \left. \frac{\partial T(I,D)}{\partial I} \right|_{I=1, D=z} \\ &= \frac{z^7}{(1-z-z^3)^2}, \end{aligned}$$

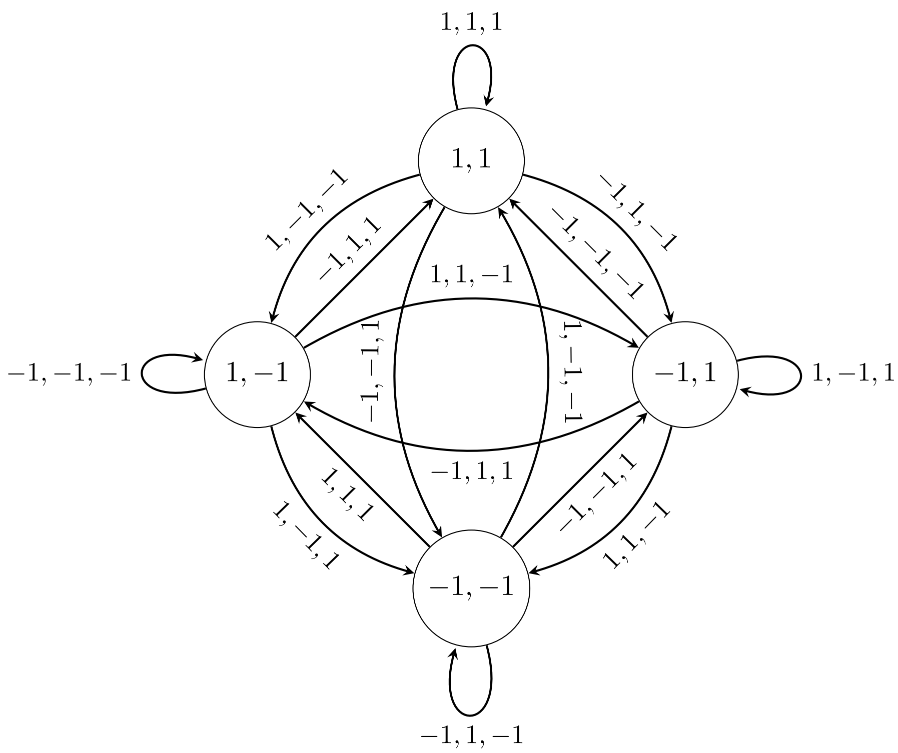
where $z = e^{-\frac{\mathcal{E}_s}{N_0}}$.

Solution 2.

(a) An implementation of the encoder will be as follows:



(b) The state diagram is shown below. We use the following terminology: the state label is x, y, where x is the "state of the even sub-sequence", i.e. contains b_{2n-2}, and y is the "state of the odd sub-sequence", i.e., contains b_{2n-1}. On the arrows, we only mark the outputs; the input required to make a particular transition is simply the next state, therefore we omitted it. Transitions are labeled with the value of $x_{3n}, x_{3n+1}, x_{3n+2}$.



(c) We use

$$P_b \leq \frac{1}{k_0} \frac{\partial T(I, D)}{\partial I} \bigg|_{I=1, D=z},$$

where $z = e^{-\frac{\mathcal{E}_s}{N_0}}$ and k_0 is the number of inputs per section of the trellis. In this problem, $k_0 = 2$. Since there are three channel symbols per two source symbols, we find that $\mathcal{E}_s = 2\mathcal{E}_b/3$.

From the state diagram we can derive the generating functions of the detour flow graph:

$$\begin{aligned} T(I, D) &= D^3 T_{-1,1} + D^2 T_{-1,-1} + D T_{1,-1} \\ T_{1,-1} &= IDT_{-1,1} + IT_{-1,-1} + ID^3 T_{1,-1} + ID^2 T_{1,1} \\ T_{-1,-1} &= I^2 DT_{-1,1} + I^2 D^2 T_{-1,-1} + I^2 DT_{1,-1} + I^2 D^2 T_{1,1} \\ T_{-1,1} &= IDT_{-1,1} + ID^2 T_{-1,-1} + IDT_{1,-1} + ID^2 T_{1,1}. \end{aligned}$$

Solving the system gives

$$T(I, D) = T_{1,1} \frac{D^2 I (D^6 I + D^5 I^2 - D^3 - D^4 I - D)}{-D^5 I^3 - D^4 I^2 + D^3 I + 2D^2 I^2 + D^2 I + DI^3 + DI^2 + DI - 1},$$

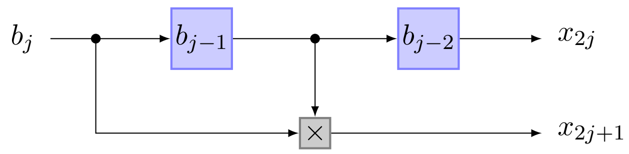
on which we can apply the formula above.

Solution 3.

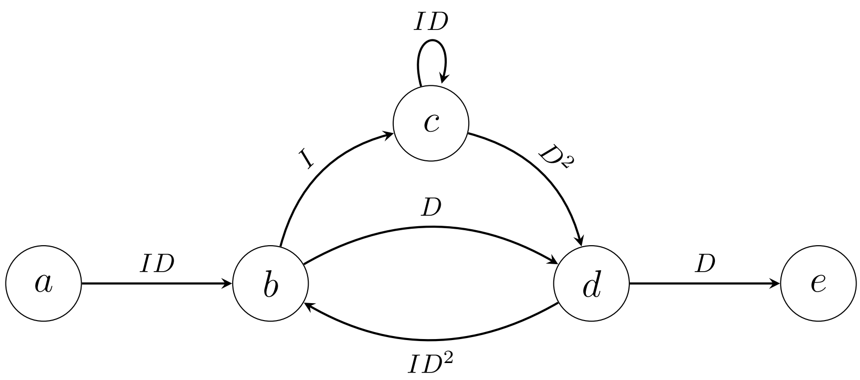
(a) Since the state is (b_{j-1}, b_{j-2}), we need two shift registers. From the finite state machine, we can derive a table that relates the state (b_{j-1}, b_{j-2}) and the current input b_j with the two outputs (x_{2j}, x_{2j+1}):

b_j	b_{j-1}	b_{j-2}	x_{2j}	x_{2j+1}
1	1	1	1	1
1	1	−1	−1	1
1	−1	1	1	−1
1	−1	−1	−1	−1
−1	1	1	1	−1
−1	1	−1	−1	−1
−1	−1	1	1	1
−1	−1	−1	−1	1

We can easily notice that the column of x_{2j} is the same as the column of b_{j-2}. Therefore, $x_{2j} = b_{j-2}$. On the other hand, we see that $x_{2j+1} = b_{j-1}$ if $b_j = 1$ and $x_{2j+1} = -b_{j-1}$ if $b_j = -1$. Therefore $x_{2j+1} = b_j \cdot b_{j-1}$, which gives us the following encoder.



(b) The detour flow graph (with respect to the all-one sequence) is given below:



We have

$$\begin{aligned} T_b &= T_a ID + T_d ID^2 \\ T_c &= T_b I + T_c ID \\ T_d &= T_c D^2 + T_b D \\ T_e &= T_d D \end{aligned}$$

The solution of this system is $T_e = T_a \frac{ID^3}{1-ID-ID^3}$. Hence,

$$\begin{aligned} P_b \leq \left.\frac{\partial T(I,D)}{\partial I}\right|_{I=1,D=z} &= \left.\frac{D^3(1-ID-ID^3) + ID^3(D+D^3)}{(1-ID-ID^3)^2}\right|_{I=1,D=z} \\ &= \frac{z^3}{(1-z-z^3)^2}, \end{aligned}$$

where $z = e^{-\frac{\mathcal{E}_b}{2N_0}}$.

Solution 4.

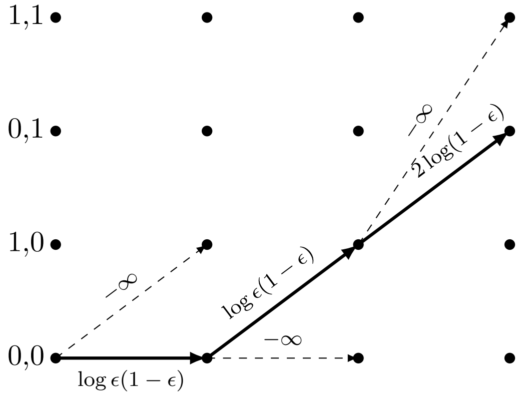
(a) The decoder is the same as in the example we have seen in Chapter 6 once the following isomorphic mapping is applied: $\{1 \to 0, -1 \to 1\}$. Figure 6.4 shows the trellis of the encoder.

(b) Given the observation $y = (y_1, \ldots, y_n)$, the ML codeword is given by $\arg\max_{x \in \mathcal{C}} p(y|x)$ where $\mathcal{C}$ represents the set of codewords (i.e., the set of all possible paths on the trellis). Alternately, the ML codeword is given by $\arg\max_{x \in \mathcal{C}} \sum_{i=1}^{n} \log p(y_i|x_i)$.

Hence, a branch metric for the BEC is

$$\log p(y_i|x_i) = \begin{cases} \log \epsilon & \text{if } y_i = ?, \\ \log(1-\epsilon) & \text{if } y_i = x_i, \\ -\infty & \text{if } y_i = 1 - x_i. \end{cases}$$

(c) Given the observation $(0, ?, ?, 1, 0, 1)$, one can compute the branch metric in the trellis. Note that we do not need to further elaborate paths with a $-\infty$ metric. The decoding results $(0, 1, 0)$.



(d) We refer to the example shown in Chapter 6, where we have the same encoder, but a different channel. We have seen that

$$P_b \leq \frac{z^5}{(1-2z)^2}.$$

To determine z we use the Bhattacharyya bound, which in our case is

$$z = \sum_{y \in \{0,1,?\}} \sqrt{P(y|1)P(y|0)} = \epsilon.$$

Thus we have the following bound:

$$P_b \leq \frac{\epsilon^5}{(1-2\epsilon)^2}.$$