SOLUTION 1.

(a) The state diagram and detour flow graph are respectively shown below:

State diagram

Detour flow graph

(b) Let $a, b, c, d, e$ respectively represent the states $(1, 1), (-1, 1), (-1, -1), (1, -1)$ and $(1, 1)$. We have

$$T_b = T_d ID + T_a ID^2$$
$$T_c = T_c ID + T_b ID^2$$
$$T_d = T_b D^2 + T_c D.$$
Substituting $T_c = T_b \frac{ID^2}{1-ID^3}$ in the third equation above,

$$T_d = T_b D^2 + T_b \frac{1-ID^3}{1-ID}$$

$$= T_b \left( D^2 + \frac{ID^3}{1-ID} \right)$$

$$= T_b \frac{D^2}{1-ID}$$

$$= T_b \alpha,$$

with $\alpha = \frac{D^2}{1-ID}$. The detour flow graph can thus be simplified to:

![Detour Flow Graph](image)

In $T_b = T_d ID + T_a ID^2$, we substitute for $T_d$ to get

$$T_b = T_a \frac{ID^2 (1-ID)}{1-ID-1ID^3}.$$

It follows that

$$T_d = T_b \frac{D^2}{1-ID} = T_a \frac{ID^4}{1-ID-1ID^3},$$

and that

$$T(I, D) = T_e = T_a \frac{ID^7}{1-ID-1ID^3}.$$

Taking the derivative yields

$$\frac{\partial T(I, D)}{\partial I} = \frac{D^7 (1-ID - ID^3) - ID^7 (-D - D^3)}{(1-ID-ID^3)^2} = \frac{D^7}{(1-ID-ID^3)^2}.$$

Therefore, we find

$$P_b \leq \left. \frac{\partial T(I, D)}{\partial I} \right|_{I=1,D=z} = \left. \frac{z^7}{(1-z-z^3)^2} \right|_{I=1,D=z},$$

where $z = e^{-\frac{\epsilon_0}{kT}}$.

**Solution 2.**

(a) An implementation of the encoder will be as follows:

![Encoder Implementation Diagram](image)
(b) The state diagram is shown below. We use the following terminology: the state label is \( x, y \), where \( x \) is the “state of the even sub-sequence”, i.e. contains \( b_{2n-2} \), and \( y \) is the “state of the odd sub-sequence”, i.e., contains \( b_{2n-1} \). On the arrows, we only mark the outputs; the input required to make a particular transition is simply the next state, therefore we omitted it. Transitions are labeled with the value of \( x_{3n}, x_{3n+1}, x_{3n+2} \).

(c) We use

\[
P_b \leq \frac{1}{k_0} \left. \frac{\partial T(I, D)}{\partial I} \right|_{I=1, D=z},
\]

where \( z = e^{-\frac{E_s}{k_0}} \) and \( k_0 \) is the number of inputs per section of the trellis. In this problem, \( k_0 = 2 \). Since there are three channel symbols per two source symbols, we find that \( E_s = 2E_b/3 \).

From the state diagram we can derive the generating symbols of the detour flow graph:

\[
T(I, D) = D^3T_{-1,1} + D^2T_{-1,-1} + DT_{1,-1} \\
T_{1,-1} = IDT_{-1,1} + IT_{-1,-1} + ID^2T_{1,-1} + ID^2T_{1,1} \\
T_{-1,-1} = I^2DT_{-1,1} + I^2D^2T_{-1,-1} + I^2DT_{1,-1} + I^2D^2T_{1,1} \\
T_{-1,1} = IDT_{-1,1} + ID^2T_{-1,-1} + IDT_{1,-1} + ID^2T_{1,1}.
\]

Solving the system gives

\[
T(I, D) = T_{1,1} \frac{D^2I(D^6I + D^5I^2 - D^3 - D^4I - D)}{-D^5I^3 - D^4I^2 + D^3I + 2D^2I^2 + D^2I + DI^3 + DI^2 + DI - 1},
\]

on which we can apply the formula above.
Solution 3.

(a) Since the state is \((b_{j-1}, b_{j-2})\), we need two shift registers. From the finite state machine, we can derive a table that relates the state \((b_{j-1}, b_{j-2})\) and the current input \(b_{j}\) with the two outputs \((x_{2j}, x_{2j+1})\):

<table>
<thead>
<tr>
<th>(b_{j})</th>
<th>(b_{j-1})</th>
<th>(b_{j-2})</th>
<th>(x_{2j})</th>
<th>(x_{2j+1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>1</td>
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<td>1</td>
</tr>
<tr>
<td>1</td>
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<tr>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

We can easily notice that the column of \(x_{2j}\) is the same as the column of \(b_{j-2}\). Therefore, \(x_{2j} = b_{j-2}\). On the other hand, we see that \(x_{2j+1} = b_{j-1}\) if \(b_{j} = 1\) and \(x_{2j+1} = -b_{j-1}\) if \(b_{j} = -1\). Therefore \(x_{2j+1} = b_{j} \cdot b_{j-1}\), which gives us the following encoder.

(b) The detour flow graph (with respect to the all-one sequence) is given below:

We have

\[
T_b = T_a ID + T_d ID^2 \\
T_c = T_b I + T_c ID \\
T_d = T_c D^2 + T_b D \\
T_e = T_d D
\]

The solution of this system is \(T_e = T_a \frac{ID^3}{1-ID-ID^3}\). Hence,

\[
P_b \leq \left. \frac{\partial T(I,D)}{\partial I} \right|_{I=1,D=z} = \left. \frac{D^3(1-ID-ID^3)+ID^3(D+D^3)}{(1-ID-ID^3)^2} \right|_{I=1,D=z}
\]

\[
= \frac{z^3}{(1-z-z^3)^2}
\]

where \(z = e^{-\frac{\epsilon_p}{T_N}}\).

4
Solution 4.

(a) The decoder is the same as in the example we have seen in Chapter 6 once the following isomorphic mapping is applied: \(\{1 \rightarrow 0, -1 \rightarrow 1\}\). Figure 6.4 shows the trellis of the encoder.

(b) Given the observation \(y = (y_1, \ldots, y_n)\), the ML codeword is given by \(\text{arg max}_{x \in \mathcal{C}} p(y|x)\) where \(\mathcal{C}\) represents the set of codewords (i.e., the set of all possible paths on the trellis). Alternately, the ML codeword is given by \(\text{arg max}_{x \in \mathcal{C}} \sum_{i=1}^{n} \log p(y_i|x_i)\).

Hence, a branch metric for the BEC is

\[
\log p(y_i|x_i) = \begin{cases}
\log \epsilon & \text{if } y_i =?, \\
\log(1 - \epsilon) & \text{if } y_i = x_i, \\
-\infty & \text{if } y_i = 1 - x_i.
\end{cases}
\]

(c) Given the observation \((0, ?, ?, 1, 0, 1)\), one can compute the branch metric in the trellis. Note that we do not need to further elaborate paths with a \(-\infty\) metric. The decoding results \((0, 1, 0)\).

(d) We refer to the example shown in Chapter 6, where we have the same encoder, but a different channel. We have seen that

\[
P_b \leq \frac{z^5}{(1 - 2z)^2}.
\]

To determine \(z\) we use the Bhattacharyya bound, which in our case is

\[
z = \sum_{y \in \{0,1,?\}} \sqrt{P(y|1)P(y|0)} = \epsilon.
\]

Thus we have the following bound:

\[
P_b \leq \frac{\epsilon^5}{(1 - 2\epsilon)^2}.
\]