SOLUTION 1. In Section 5.3, it is shown that the power spectral density is

$$S_X(f) = \frac{|\psi_{\mathcal{F}}(f)|^2}{T} \sum_k K_X[k] \exp(-\mathrm{j}2\pi k f T),$$

where $K_X[k]$ is the auto-covariance of X_i and $\psi_{\mathcal{F}}(f)$ is the Fourier transform of $\psi(t)$.

Because $\{X_i\}_{i=-\infty}^{\infty}$ are i.i.d. and have zero-mean,

$$K_X[k] = \mathbb{E}[X_{i+k}X_i^*] = \mathcal{E}\mathbb{1}\{k=0\},$$

so

$$S_X(f) = \mathcal{E}\frac{|\psi_{\mathcal{F}}(f)|^2}{T}.$$

Moreover,

$$\begin{aligned}
\psi_{\mathcal{F}}(f) &= \int_{-\infty}^{\infty} \psi(t) e^{-\mathrm{j}2\pi f t}\, dt \\
&= \frac{1}{\sqrt{T}} \int_0^{\frac{T}{2}} e^{-\mathrm{j}2\pi f t}\, dt - \frac{1}{\sqrt{T}} \int_{\frac{T}{2}}^{T} e^{-\mathrm{j}2\pi f t}\, dt \\
&= \frac{\mathrm{j}}{2\pi f\sqrt{T}} \left(e^{-\mathrm{j}2\pi f \frac{T}{2}} - 1 - e^{-\mathrm{j}2\pi f T} + e^{-\mathrm{j}2\pi f \frac{T}{2}} \right) \\
&= \frac{\mathrm{j}}{2\pi f\sqrt{T}} e^{-\mathrm{j}2\pi f \frac{T}{2}} \left(2 - e^{\mathrm{j}2\pi f \frac{T}{2}} - e^{-\mathrm{j}2\pi f \frac{T}{2}} \right) \\
&= \frac{\mathrm{j}}{2\pi f\sqrt{T}} e^{-\mathrm{j}2\pi f \frac{T}{2}} (2 - 2\cos(\pi f T)) \\
&= \frac{\mathrm{j}}{2\pi f\sqrt{T}} e^{-\mathrm{j}2\pi f \frac{T}{2}} 4\sin^2\left(\pi f \frac{T}{2}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
S_X(f) &= \mathcal{E}\frac{16\sin^4(\pi f \frac{T}{2})}{4\pi^2 f^2 T^2} \\
&= \mathcal{E}\left(\pi f \frac{T}{2}\right)^2 \operatorname{sinc}^4\left(f\frac{T}{2}\right).
\end{aligned}$$

SOLUTION 2.

(a) When $i = j$, $\mathbb{E}[X_i X_j]$ equals

$$\mathbb{E}[X_i^2] = \mathbb{E}[1] = 1.$$

Remember that the B_i are i.i.d. Bernoulli($\frac{1}{2}$) random variables. Hence, we find immediately

$$\begin{aligned}
K_X[1] = \mathbb{E}[X_{2n}X_{2n+1}] &= \mathbb{E}[B_n B_{n-2} B_n B_{n-1} B_{n-2}] \\
&= \mathbb{E}[B_n^2 B_{n-1} B_{n-2}^2] \\
&= \mathbb{E}[B_{n-1}] = 0,
\end{aligned}$$

and also

$$K_X[2] = \mathbb{E}[X_{2n}X_{2n+2}] = \mathbb{E}[B_n B_{n-2} B_{n+1} B_{n-1}]$$
$$= \mathbb{E}[B_n]\,\mathbb{E}[B_{n-2}]\,\mathbb{E}[B_{n+1}]\,\mathbb{E}[B_{n-1}] = 0.$$

By continuing this argument we find

$$K_X[i] = \mathbb{1}\{i = 0\}.$$

Hence,

$$S_X(f) = \frac{\mathcal{E}_s}{T_s}|\psi_{\mathcal{F}}(f)|^2.$$

This means that by choosing $\psi(t)$ appropriately, we can control the bandwidth consumption of our communications scheme.

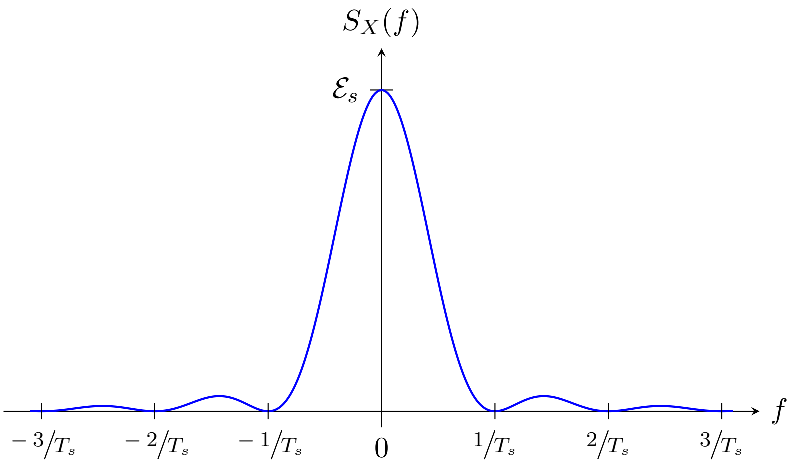
(b) We know that

$$|\psi_{\mathcal{F}}(f)|^2 = T_s \operatorname{sinc}^2(T_s f).$$

It follows that

$$S_X(f) = \mathcal{E}_s \operatorname{sinc}^2(T_s f).$$

A plot of $S_X(f)$ is shown below:

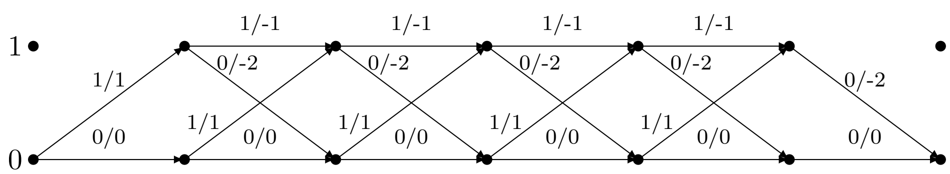


Solution 3.

(a)

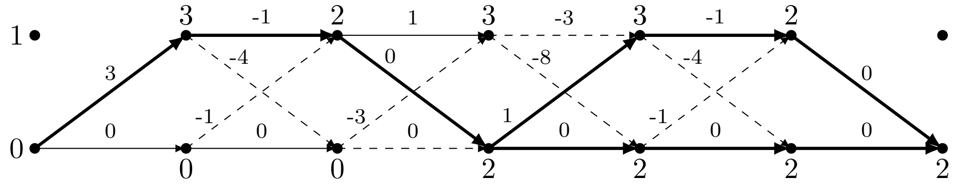
$$X_i = B_i - 2B_{i-1}$$

From this, we can draw the following trellis:



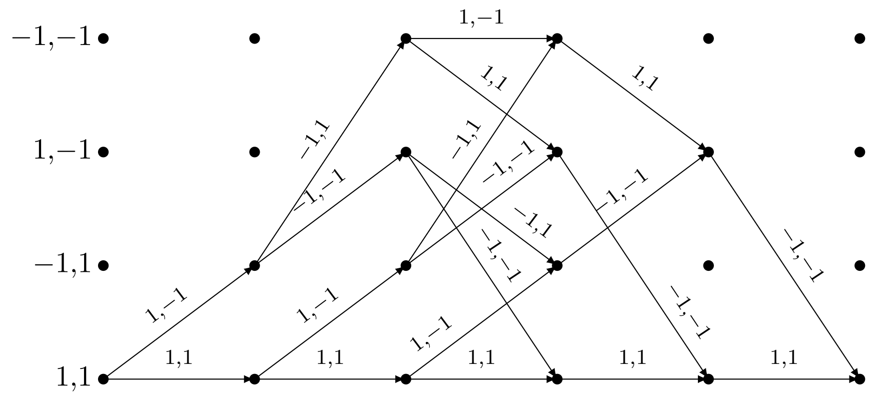
(b) We have $Y = X + Z$, where $Z = (Z_1, \ldots, Z_6)$ is a sequence of i.i.d. components with $Z_i \sim \mathcal{N}(0, \sigma^2)$. Our maximum likelihood decoder is a minimum distance decoder. We have to minimize $\|y - x\|^2$ or equivalently, maximize $2\langle y, x\rangle - \|x\|^2$. We thus have $f(x, y) = \sum_{i=1}^{6}(2y_i x_i - x_i^2)$ whose maximization with respect to x leads to a maximum likelihood decision on X.

(c) We label our trellis with the edge metric $2y_ix_i - x_i^2$ and then trace back the decoding path.

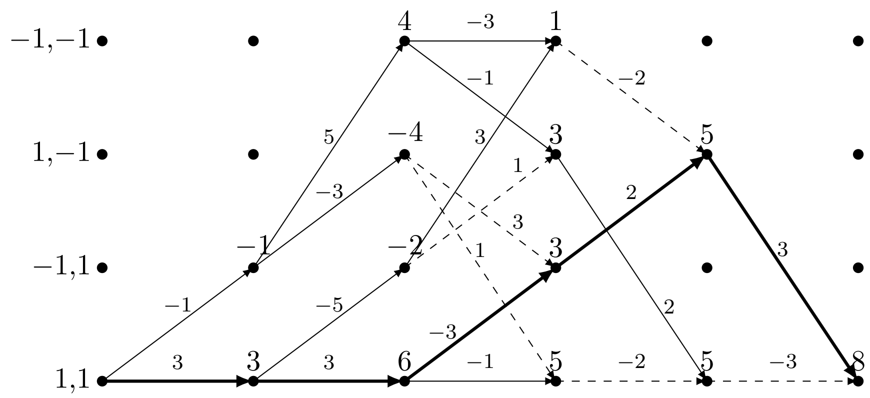


We see that the two sequences 1,1,0,0,0 and 1,1,0,1,1 are equally likely, so the decoder would choose either of the two.

Solution 4. The trellis representing the encoder is shown below:



We display the diagram labeled with edge-metric according to the received sequence and state-metric of the survivor path. We also indicate the survivor paths and the decoding path.



From the figure we can read the decoded sequence $1, 1, -1, 1, 1$.

Solution 5. The output of encoder (a) is

$$\begin{aligned} T(\bar{x}_{2j-1}) &= T(\bar{b}_j + \bar{b}_{j-2}) = T(\bar{b}_j)T(\bar{b}_{j-2}) = b_jb_{j-2} \\ T(\bar{x}_{2j}) &= T(\bar{b}_j + \bar{b}_{j-1} + \bar{b}_{j-2}) = T(\bar{b}_j)T(\bar{b}_{j-1})T(\bar{b}_{j-2}) = b_jb_{j-1}b_{j-2}, \end{aligned}$$

which is identical to the output of encoder (b).