SOLUTION 1.

(a) We have $\tri_F(f) = \rect_F^2(f) = \sinc^2(f)$ because $\tri(t) = (\rect \ast \rect)(t)$. Now since $\psi(t) = \sinc(t) \cdot \tri(t)$, we have

$$\psi_F(f) = (\sinc_F \ast \tri_F)(f) = (\rect \ast \sinc^2)(f)$$

$$= \int_{-\infty}^{+\infty} \rect(f - u) \sinc^2(u) du = \int_{f - \frac{1}{2}}^{f + \frac{1}{2}} \sinc^2(u) du = a(f).$$

On the other hand, we have $E[X_i] = 0$ and

$$K_X[k] = E[X_{i+k}X_i^*] = \mathcal{E} \{k = 0\}.$$

Thus,

$$S_X(f) = |\psi_F(f)|^2 \sum_k K_X[k] e^{-j2\pi kf} = \mathcal{E} a^2(f).$$

Now since $h(t) = \delta(t) - 2\delta(t - 1/4) + \delta(t - 1/2)$, we have

$$h_F(f) = 1 - 2e^{-j\frac{\pi f}{2}} + e^{-j\pi f} = \left(1 - e^{-j\frac{\pi f}{2}}\right)^2.$$ 

Therefore,

$$S_Y(f) = S_X(f) \cdot |h_F(f)|^2 = \mathcal{E} a^2(f) \cdot \left|1 - e^{-j\frac{\pi f}{2}}\right|^4.$$ 

Since $a(f) > 0$ for every $f \in \mathbb{R}$, we have $S_Y(f) = 0$ if and only if $1 - e^{-j\frac{\pi f}{2}} = 0$. Therefore,

$$S_Y(f) = 0 \iff f = 4m \text{ for some } m \in \mathbb{Z}.$$

(b) We have

$$E[X_i^2] = s^2 \left(E[D_i^2] + 2\alpha E[D_i D_{i-4}] + \alpha^2 E[D_{i-4}^2]\right) = s^2 (1 + 0 + \alpha^2) = s^2 (1 + \alpha^2).$$

Hence,

$$s = \pm \sqrt{\frac{\mathcal{E}}{1 + \alpha^2}}.$$ 

We still have $E[X_i] = 0$. However,

$$K_X[k] = E[X_{i+k}X_i^*]$$

$$= s^2 \left(E[D_{i+k}D_i] + \alpha E[D_{i+k} D_{i-4}] + \alpha E[D_{i+k-4} D_i] + \alpha^2 E[D_{i+k-4} D_{i-4}]\right)$$

$$= s^2 \left((1 + \alpha^2) \mathbb{1}\{k = 0\} + \alpha \mathbb{1}\{k = -4\} + \alpha \mathbb{1}\{k = 4\}\right)$$

$$= \mathcal{E} \left(\mathbb{1}\{k = 0\} + \frac{\alpha}{1 + \alpha^2} \mathbb{1}\{k = -4\} + \frac{\alpha}{1 + \alpha^2} \mathbb{1}\{k = 4\}\right).$$
Therefore,

\[ S_X(f) = |\psi_F(f)|^2 \sum_k K_X[k] e^{-j2\pi k f} \]

\[ = |\psi_F(f)|^2 \left( 1 + \frac{\alpha}{1 + \alpha^2} e^{j8\pi f} + \frac{\alpha}{1 + \alpha^2} e^{-j8\pi f} \right) \]

\[ = E \cdot a^2(f) \left( 1 + \frac{2\alpha}{1 + \alpha^2} \cos(8\pi f) \right), \]

and

\[ S_Y(f) = S_X(f) \cdot |h_F(f)|^2 = E a^2(f) \left( 1 + \frac{2\alpha}{1 + \alpha^2} \cos(8\pi f) \right) \cdot \left| 1 - e^{-j\frac{\pi f}{2}} \right|^4. \]

We have several cases:

- If $|\alpha| \neq 1$, we have $\left| \frac{2\alpha}{1 + \alpha^2} \right| < 1$ and so $1 + \frac{2\alpha}{1 + \alpha^2} \cos(8\pi f) \neq 0$ for every $f \in \mathbb{R}$.
  Hence,

  \[ S_Y(f) = 0 \iff 1 - e^{-j\frac{\pi f}{2}} = 0 \iff f = 4m \text{ for some } m \in \mathbb{Z}. \]

- If $\alpha = -1$, we have

  \[ S_Y(f) = S_X(f) \cdot |h_F(f)|^2 = E a^2(f) (1 - \cos(8\pi f)) \cdot \left| 1 - e^{-j\frac{\pi f}{2}} \right|^4. \]

  Hence,

  \[ S_Y(f) = 0 \iff 1 - e^{-j\frac{\pi f}{2}} = 0 \text{ or } 1 - \cos(8\pi f) = 0 \iff \begin{array}{l}
  f = 4m \text{ or } f = \frac{m}{4} \text{ for some } m \in \mathbb{Z} \\
  \end{array} \iff \begin{array}{l}
  f = \frac{m}{4} \text{ for some } m \in \mathbb{Z}. \\
  \end{array} \]

- If $\alpha = 1$, we have

  \[ S_Y(f) = S_X(f) \cdot |h_F(f)|^2 = E a^2(f) (1 + \cos(8\pi f)) \cdot \left| 1 - e^{-j\frac{\pi f}{2}} \right|^4. \]

  Hence,

  \[ S_Y(f) = 0 \iff 1 - e^{-j\frac{\pi f}{2}} = 0 \text{ or } 1 + \cos(8\pi f) = 0 \iff \begin{array}{l}
  f = 4m \text{ or } f = \frac{2m + 1}{8} \text{ for some } m \in \mathbb{Z}. \\
  \end{array} \]

(c) Since $D_i^2 = D_{i-1}^2 = 1$, we have $X_i = s(D_i + D_{i-1})$. Hence

\[ \mathbb{E}[X_i^2] = s^2 \left( \mathbb{E}[D_i^2] + 2\mathbb{E}[D_i D_{i-1}] + \mathbb{E}[D_{i-1}^2] \right) = s^2(1 + 0 + 1) = 2s^2. \]

and

\[ s = \pm \sqrt{\frac{E}{2}}. \]
We still have \( E[X_1] = 0 \). However,

\[
K_X[k] = E[X_{i+k}X_i^*] = s^2 (E[D_{i+k}D_i] + E[D_{i+k}D_{i-1}] + E[D_{i+k-1}D_i] + E[D_{i+k-1}D_{i-1}]) \\
= s^2 (21\{k = 0\} + 1\{k = -1\} + 1\{k = 1\}) \\
= \mathcal{E} \left( 1\{k = 0\} + \frac{1}{2}1\{k = -1\} + \frac{1}{2}1\{k = 1\} \right).
\]

Hence,

\[
S_X(f) = |\psi_F(f)|^2 \sum_k K_X[k] e^{-j2\pi kf} \\
= \mathcal{E} \cdot a^2(f) \left( 1 + \frac{1}{2}e^{j2\pi f} + \frac{1}{2}e^{-j2\pi f} \right) \\
= \mathcal{E} \cdot a^2(f) (1 + \cos(2\pi f)),
\]

and

\[
S_Y(f) = S_X(f) \cdot |h_F(f)|^2 = \mathcal{E}a^2(f) (1 + \cos(2\pi f)) \cdot \left| 1 - e^{-j2\pi f} \right|^4.
\]

Therefore,

\[
S_Y(f) = 0 \iff 1 - e^{-j2\pi f} = 0 \text{ or } 1 + \cos(2\pi f) = 0 \\
\iff f = 4m \text{ or } f = \frac{2m+1}{2} \text{ for some } m \in \mathbb{Z}.
\]

(d) Since \( 1 \notin \{4m : m \in \mathbb{Z}\} \) and \( 1 \notin \{\frac{2m+1}{2} : m \in \mathbb{Z}\} \), we have \( S_Y(1) \neq 0 \) for the method in (c).

On the other hand, since \( 1 \notin \{4m : m \in \mathbb{Z}\} \), \( 1 \notin \{\frac{2m+1}{2} : m \in \mathbb{Z}\} \) and \( 1 \in \{\frac{m}{4} : m \in \mathbb{Z}\} \), the only value of \( \alpha \) in (b) for which we have \( S_Y(1) = 0 \) is \( \alpha = -1 \).

**Solution 2.**

(a) Since \( w_E(t) = \sum_j X_j \text{sinc}(t - j) \), we have \( w_{E,F}(f) = \sum_j X_j \text{rect}(f)e^{-j2\pi jf} \). Hence

\[
w_{E,F}(f) = 0 \text{ for } f \notin (-\frac{1}{2}, \frac{1}{2}).
\]

Now since \( w_F(f) = \frac{1}{\sqrt{2}}w_{E,F}(f - f_c) \) for \( f > 0 \), the equality \( w_F(f_c - f) = w_F(f_c + f)^* \) implies that \( w_{E,F}(-f) = w_{E,F}(f)^* \) for \( f \in (-\frac{1}{2}, \frac{1}{2}) \). But \( w_{E,F}(f) = 0 \) for \( f \notin (-\frac{1}{2}, \frac{1}{2}) \). Therefore, \( w_{E,F}(-f) = w_{E,F}(f)^* \) for every \( f \in \mathbb{R} \), which implies that \( w_E(t) \) is a real signal, i.e., \( X_j \in \{+1, -1\} \) for all \( j \in \mathbb{Z} \).

(b) Since \( w_F(f) = \frac{1}{\sqrt{2}}w_{E,F}(f - f_c) \) for \( f > 0 \), the equality \( w_F(f_c - f) = -w_F(f_c + f)^* \) implies that \( w_{E,F}(-f) = -w_{E,F}(f)^* \) for \( f \in (-\frac{1}{2}, \frac{1}{2}) \). But \( w_{E,F}(f) = 0 \) for \( f \notin (-\frac{1}{2}, \frac{1}{2}) \). Therefore, \( w_{E,F}(-f) = -w_{E,F}(f)^* \) for every \( f \in \mathbb{R} \), which implies that \( w_E(t) \) is a pure-imaginary signal, i.e., \( X_j \in \{+j, -j\} \) for all \( j \in \mathbb{Z} \).

(c) If \( \theta = 0 \),

\[
\mathbb{R}\{R_E(t)\} = A \cdot \mathbb{R}\{w_E(t)\} + N_R(t), \\
\mathbb{S}\{R_E(t)\} = A \cdot \mathbb{S}\{w_E(t)\} + N_I(t),
\]

3
where $N_R(t)$ and $N_I(t)$ are independent white Gaussian noise processes of power spectral density $\frac{N_0}{2}$.

A sufficient statistic to estimate $X_j$ from the received signal is obtained by computing the (complex-valued) inner products

$$Y_j = \langle R_E(t), \text{sinc}(t - j) \rangle,$$

or equivalently, pairs of real-valued inner products

$$Y_{1,j} = \langle \text{Re}\{R_E(t)\}, \text{sinc}(t - j) \rangle \quad \text{and} \quad Y_{2,j} = \langle \text{Im}\{R_E(t)\}, \text{sinc}(t - j) \rangle.$$

To this end, one in principle has to filter the outputs of the down-converter using matched filters of impulse response $\text{sinc}^*(-t)$ and sample the outputs of the filters at times $t = j, j \in \mathbb{Z}$. However, a filter with impulse response $\text{sinc}^*(-t)$ is nothing but a low-pass filter with frequency response $1\{ -\frac{1}{2} \leq f \leq \frac{1}{2} \}$ which is already included in the down-converter. Thus, it is sufficient to sample the output of the down-converters directly to obtain the desired sufficient statistics.

\[
\sqrt{2} \cos(2\pi f_c t + \theta) \\
R(t) \\
\times \\
\sqrt{2} \sin(2\pi f_c t + \theta)
\]

\[
\begin{array}{c}
\text{1}\{-B \leq f \leq B\} \\
\text{Re}\{R_E(t)\} \\
t = j \\
Y_{1,j}
\end{array}
\quad
\begin{array}{c}
\text{1}\{-B \leq f \leq B\} \\
\text{Im}\{R_E(t)\} \\
t = j \\
Y_{2,j}
\end{array}
\]

(d) We have the following hypothesis testing problem:

under $H = i : \quad Y = c_i + Z,$

where $Z \sim \mathcal{N}(0, \frac{N_0}{2} I_2)$ and $c_1 = [A, 0]$, $c_2 = [0, A]$, $c_3 = [-A, 0]$, and $c_4 = [0, -A]$.

For an AWGN setting, the ML decision rule will be the minimum distance decision rule with the following decision regions:

This is a 4-PSK constellation and the probability of error of an ML decoder for such a constellation is

$$P_e = 2Q\left(\frac{A}{\sqrt{N_0}}\right) - Q\left(\frac{A}{\sqrt{N_0}}\right)^2.$$
Since the decision regions in (d) do not depend on $A$, we do not change anything in the decoder of (d), even if $A$ were unknown to the receiver. The average probability of error is given by

$$P_e = \mathbb{E} \left[ 2Q \left( \frac{A}{\sqrt{N_0}} \right) - Q \left( \frac{A}{\sqrt{N_0}} \right)^2 \right]$$

$$= \frac{1}{2} \left( 2Q \left( \frac{1}{\sqrt{2N_0}} \right) - Q \left( \frac{1}{\sqrt{2N_0}} \right)^2 \right) + \frac{1}{2} \left( 2Q \left( \sqrt{\frac{3}{2N_0}} \right) - Q \left( \sqrt{\frac{3}{2N_0}} \right)^2 \right)$$

$$= Q \left( \frac{1}{\sqrt{2N_0}} \right) - \frac{1}{2} Q \left( \frac{1}{\sqrt{2N_0}} \right)^2 + Q \left( \sqrt{\frac{3}{2N_0}} \right) - \frac{1}{2} Q \left( \sqrt{\frac{3}{2N_0}} \right)^2.$$

Using the trigonometric identity $\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)$ we can see that the output of the top modulator, in presence of the phase difference, is

$$R(t) \cos(\theta) \times \sqrt{2} \cos(2\pi f_c t) - R(t) \sin(\theta) \times \sqrt{2} \sin(2\pi f_c t).$$

Thus, as the low-pass filter is a linear system, the output of the top low-pass filter is:

$$\mathcal{R}\{R_E(t)\} = \mathcal{R}\{w_E(t)\} \cos(\theta) + \mathcal{I}\{w_E(t)\} \sin(\theta) + \cos(\theta) N_R(t) + \sin(\theta) N_I(t).$$

Similarly, we can show that the output of the bottom low-pass filter is:

$$\mathcal{I}\{R_E(t)\} = \mathcal{R}\{w_E(t)\} \cos(\theta) - \mathcal{I}\{w_E(t)\} \sin(\theta) + \cos(\theta) N_I(t) - \sin(\theta) N_R(t).$$

Therefore, the observable $Y = [Y_1, Y_2]$ (under $H = i$) is now equal to

$$Y = R_{\theta} c_i + R_{\theta} Z$$

where

$$R_{\theta} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

is the rotation matrix, the codewords $c_i$ are as in part (d), and $Z \sim \mathcal{N}(0, \frac{N_0}{2} I_2)$. Moreover, we know that $V = R_{\theta} Z$ has the same statistics as $Z$. Thus, we can write the observable $Y$ as

$$Y = R_{\theta} c_i + V$$

with $V \sim \mathcal{N}(0, \frac{N_0}{2} I_2)$. 

\[\text{Diagram:} \]

```
Y_2
<p>| | |</p>
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|   | R_2
|   |   |
|   |   |
Y_1
<p>| | |</p>
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</table>
|   | R_1
|   |   |
|   |   |
R_3
|   |   |
|   |   |
|   |   |
R_4
```
Using the above diagram, we can see that the probability of correct guess of the receiver, conditioned on a particular value of $A$, is

$$
P_c = Q \left( \frac{A \sin(\theta - \frac{\pi}{4})}{\sqrt{N_0/2}} \right) Q \left( -\frac{A \cos(\theta - \frac{\pi}{4})}{\sqrt{N_0/2}} \right)
$$

$$
= \left( 1 - Q \left( \frac{A \sin(\frac{\pi}{4} - \theta)}{\sqrt{N_0/2}} \right) \right) \left( 1 - Q \left( \frac{A \cos(\frac{\pi}{4} - \theta)}{\sqrt{N_0/2}} \right) \right)
$$

$$
= 1 - Q \left( \frac{A \sin(\frac{\pi}{4} - \theta)}{\sqrt{N_0/2}} \right) - Q \left( \frac{A \cos(\frac{\pi}{4} - \theta)}{\sqrt{N_0/2}} \right) + Q \left( \frac{A \sin(\frac{\pi}{4} - \theta)}{\sqrt{N_0/2}} \right) Q \left( \frac{A \cos(\frac{\pi}{4} - \theta)}{\sqrt{N_0/2}} \right).
$$

Therefore, the average probability of error is given by

$$
P_e = E \left[ Q \left( \frac{A \sin(\frac{\pi}{4} - \theta)}{\sqrt{N_0/2}} \right) + Q \left( \frac{A \cos(\frac{\pi}{4} - \theta)}{\sqrt{N_0/2}} \right) - Q \left( \frac{A \sin(\frac{\pi}{4} - \theta)}{\sqrt{N_0/2}} \right) Q \left( \frac{A \cos(\frac{\pi}{4} - \theta)}{\sqrt{N_0/2}} \right) \right]
$$

$$
= \frac{1}{2} \left( Q \left( \frac{\sin(\frac{\pi}{4} - \theta)}{\sqrt{N_0}} \right) + Q \left( \frac{\cos(\frac{\pi}{4} - \theta)}{\sqrt{N_0}} \right) - Q \left( \frac{\sin(\frac{\pi}{4} - \theta)}{\sqrt{N_0}} \right) Q \left( \frac{\cos(\frac{\pi}{4} - \theta)}{\sqrt{N_0}} \right) \right)
$$

$$
+ \frac{1}{2} \left( Q \left( \frac{\sqrt{3}\sin(\frac{\pi}{4} - \theta)}{\sqrt{N_0}} \right) + Q \left( \frac{\sqrt{3}\cos(\frac{\pi}{4} - \theta)}{\sqrt{N_0}} \right) - Q \left( \frac{\sqrt{3}\sin(\frac{\pi}{4} - \theta)}{\sqrt{N_0}} \right) Q \left( \frac{\sqrt{3}\cos(\frac{\pi}{4} - \theta)}{\sqrt{N_0}} \right) \right).
$$

**Solution 3.**

(a) The state diagram and detour flow graph are shown here. The states are labeled as $(b_{j-1}, b_{j-2})$ and the transitions with $b_j/x_3j, x_{3j+1}, x_{3j+2}$.

```
-1/1, -1, 1
  
-1, -1
  
1/1, -1, -1
  
-1, 1
  
1, -1
  
-1/1, 1, -1
  
-1/1, 1, -1
  
1, 1
  
1/1, 1, 1
```
(b) The output to \((1, -1, -1, 1, 1)\) is \((1, 1, 1, -1, -1, -1, 1, 1, -1, 1, -1, 1, -1, 1, 1)\).

(c) The Bhattacharyya bound is given by
\[
\begin{align*}
  z &= \sum_y \sqrt{\mathbb{P}_{Y|X}(y|1)\mathbb{P}_{Y|X}(y|1)} \\
  &= \sqrt{\mathbb{P}_{Y|X}(1|1)\mathbb{P}_{Y|X}(1|1)} + \sqrt{\mathbb{P}_{Y|X}(1|1)\mathbb{P}_{Y|X}(1|1)} + \sqrt{\mathbb{P}_{Y|X}(-1|1)\mathbb{P}_{Y|X}(-1|1)} \\
  &= \sqrt{(1 - \epsilon)^2 + \epsilon^2(1 - \epsilon)^2 + \epsilon^2(1 - \epsilon)^2} \\
  &= \sqrt{2\epsilon(1 - \epsilon) + \epsilon^2}.
\end{align*}
\]

(d) Let us relabel the states in the detour flow graph:

We have:
\[
\begin{align*}
  T_l &= ID^3 T_s + ID T_r, \\
  T_r &= D^2 T_l + D^2 T_t, \\
  T_t &= ID T_l + ID T_t,
\end{align*}
\]
and
\[
T_e = D^2 T_r.
\]

We have
\[
T_l + T_t = ID^3 T_s + ID T_r + T_t = ID^3 T_s + (ID^3 + ID)(T_l + T_t),
\]
hence,

\[ T_i + T_t = \frac{ID^3}{1 - ID - ID^3} T_s. \]

Therefore,

\[ T_e = D^2 T_r = D^4 (T_i + T_t) = \frac{ID^7}{1 - ID - ID^3}. \]

The generating function of \( a(i, d) \) is

\[ T(I, D) = \frac{T_e}{T_s} = \frac{ID^7}{1 - ID - ID^3}. \]

We have

\[ \frac{\partial T}{\partial I} (I, D) = \frac{D^7 (1 - ID - ID^3) - (D - D^3) ID^7}{(1 - ID - ID^3)^2} = \frac{D^7}{(1 - ID - ID^3)^2}. \]

We conclude that the bit error probability is upper bounded as follows:

\[ P_e \leq \frac{\partial T}{\partial I} (1, z) = \frac{z^7}{(1 - z - z^3)^2}. \]

(e) Given the observation \( y = (y_0, \ldots, y_{3n-1}) \), the ML codeword is given by

\[ \arg \max_{x \in C} p(y|x), \]

where \( C \) represents the set of codewords (i.e., the set of all possible paths on the trellis). Alternately, the ML codeword is given by

\[ \arg \max_{x \in C} \sum_{i=0}^{3n-1} \log p(y_i|x_i). \]

Hence, a branch metric for the Viterbi decoder is

\[ \log p(y_i|x_i) = \begin{cases} \log(1 - \epsilon) & \text{if } y_i = x_i, \\ \log \left( \frac{\epsilon}{2} \right) & \text{if } y_i \neq x_i. \end{cases} \]

The decoder chooses the path with the largest metric.

(f) The trellis representing the encoder is shown below:
We display the diagram labeled with edge-metric according to the received sequence and state-metric of the survivor path. We also indicate the survivor paths and the decoding path.

From the figure we can read the decoded sequence $1, -1, -1, 1, 1$. 

\[\begin{align*}
-1, -1 & \quad \cdots \quad -u - 5v - 2u - v - 5u - 4v - 3v \\
1, -1 & \quad \cdots \\
-1, 1 & \\
1, 1 & \quad \cdots \quad -2u - v - 2u - 4v - 5u - 6v - 11u - 4v
\end{align*}\]