**Exercise 1** Dirac’s notation for vectors and matrices

(a) If $|w\rangle$ is a vector and $\alpha$ is a scalar, then

$$ (\alpha |w\rangle)^\dagger = \langle w| \alpha^* = \alpha^* \langle w| $$

(you can check this in components). Moreover, we have linearity:

$$ (\alpha |v\rangle + \beta |w\rangle)^\dagger = (\alpha |v\rangle)^\dagger + (\beta |w\rangle)^\dagger. $$

Then we get

$$ \langle v| = (|v\rangle)^\dagger = (v_1 \langle e_1| + v_2 \langle e_2| + \cdots + v_N \langle e_N|)^\dagger $$

(b) If $\langle v| = \sum_{i=1}^{N} v_i^* \langle e_i|$ and $|w\rangle = \sum_{j=1}^{N} w_j |e_j\rangle$, then

$$ \langle v|w\rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} v_i^* w_j \langle e_i|e_j\rangle $$

$$ = \sum_{i=1}^{N} \sum_{j=1}^{N} v_i^* w_j \delta_{ij} $$

$$ = \sum_{i=1}^{N} v_i^* w_i. $$

(c) Same method provided in (a).

(d) For $\vec{v} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, we have $||\vec{v}||^2 = \vec{v}^T \cdot \vec{v}$, so $||\vec{v}||^2 = \alpha^* \alpha + \beta^* \beta$. On the other hand, $\langle v|v\rangle = \alpha^* \alpha + \beta^* \beta$ also by (b).

(e) Write the ket-bra $\langle e_k| A |e_l\rangle$ as a matrix to realize that
Thus,

$$A = \sum_{k,l} a_{kl} |e_k\rangle \langle e_l|.$$ 

So,

$$\langle e_i | A | e_j \rangle = \sum_{l,l} a_{kl} \langle e_i | e_k \rangle \langle e_l | e_j \rangle = \sum_{l,l} a_{kl} \delta_{ik} \delta_{lj} = a_{ij}.$$ 

(f) From the beginning of point (e), we have

$$I = \sum_{i=1}^{N} |e_i\rangle \langle e_i|.$$ 

Indeed, $|e_i\rangle \langle e_i|$ is the matrix with 1 at the $i$-th row and $i$-th column and zeros elsewhere. This is called the closure relation.

(g) First note that the closure relation is valid for any orthonormal basis. Indeed, if $\{|\varphi_i\rangle\}_{i=1...N}$ are orthonormal, there exists a unitary basis change (a “rotation”) such that

$$|\varphi_i\rangle = U |e_i\rangle,$$
$$\langle \varphi_i| = \langle e_i| U^\dagger.$$ 

Then from $I = \sum_{i=1}^{N} |e_i\rangle \langle e_i|$ we get:

$$UIU^\dagger = \sum_{i=1}^{N} U |e_i\rangle \langle e_i| U^\dagger$$
$$I = \sum_{i=1}^{N} |\varphi_i\rangle \langle \varphi_i|.$$ 

Now, from $\alpha_i |\varphi_i\rangle = A |\varphi_i\rangle$ we get

$$\sum_{i=1}^{N} \alpha_i |\varphi_i\rangle \langle \varphi_i| = \sum_{i=1}^{N} A |\varphi_i\rangle \langle \varphi_i| = A \sum_{i=1}^{N} |\varphi_i\rangle \langle \varphi_i| = AI = A.$$ 

Exercise 2 Tensor Product in Dirac’s notation

(a) By distributivity of the tensor product (first two properties), it follows that:

$$|v\rangle_1 \otimes |w\rangle_2 = \left( \sum_{i=1}^{N} v_i |e_i\rangle_1 \right) \otimes \left( \sum_{j=1}^{M} w_j |f_j\rangle_2 \right) = \sum_{i=1}^{N} \sum_{j=1}^{M} v_i w_j |e_i\rangle_1 \otimes |f_j\rangle_2.$$
(b) Take two vectors $|e_i, f_j\rangle$ and $|e_k, f_l\rangle$ of $H_1 \otimes H_2$. Then by definition of the inner product:

$$\langle e_k, f_l | e_i, f_j \rangle = \langle e_k | e_i \rangle \langle f_l | f_j \rangle = \delta_{kl} \cdot \delta_{ij} = \delta_{(k,l);(i,j)}.$$ 

So this equals one if and only if $(k,l) = (i,j)$ and zero otherwise. This means that

$$\{ |e_i, f_j\rangle ; i = 1 \ldots N; j = 1 \ldots N \}$$ 

is an orthonormal basis of $H_1 \otimes H_2$. The dimension equals the number of basis vectors, so is $NM$, the product of $\dim H_1$ and $\dim H_2$.

(c) We apply the definition

$$A \otimes B |\Psi\rangle = \sum_{i,j} \psi_{ij} A |e_i\rangle_1 \otimes B |f_j\rangle_2$$


to $|\Psi\rangle = |e_k, f_l\rangle$. So $\psi_{ij} = 1$ for $(i,j) = (k,l)$ and 0 otherwise. This means:

$$A \otimes B |e_k, f_l\rangle = A |e_k\rangle \otimes B |f_l\rangle$$

and multiplying by $\langle e_i, f_j |$, we find:

$$\langle e_i, f_j | A \otimes B |e_k, f_l\rangle = \left( \langle e_i | \otimes \langle f_j | \right) \left( A |e_k\rangle \otimes B |f_l\rangle \right)$$

$$= \langle e_i | A |e_k\rangle \langle f_j | B |f_l\rangle$$

$$= a_{ik} b_{jl}.$$ 