Exercise 1 Production of Bell states

a) Check the following identity using Dirac's notation:

$$|B_{xy}\rangle = (CNOT)(H \otimes I)|x\rangle \otimes |y\rangle.$$  

where $x, y \in \{0, 1\}$ and $|B_{xy}\rangle$ are the Bell states.

b) Represent the corresponding circuit.

c) Represent the circuit corresponding to the inverse identity:

$$|x\rangle \otimes |y\rangle = (H \otimes I)(CNOT)|B_{xy}\rangle$$

Exercise 2 Construction of a multi-control-$U$.

Verify that the multi-control-$U$:

\[
\begin{array}{c}
|c_1\rangle \\
|c_2\rangle \\
|c_3\rangle \\
|t\rangle \\
\end{array}
\rightarrow \begin{array}{c}
|c_1\rangle \\
|c_2\rangle \\
|c_3\rangle \\
U^{c_1c_2c_3}|t\rangle \\
\end{array}
\]

can be realized with the Toffoli gate (control-control-NOT) a simple simple control-$U$.

\[
\begin{array}{c}
|c_1\rangle \\
|c_2\rangle \\
|c_3\rangle \\
|0\rangle \\
|0\rangle \\
|c_4\rangle \\
\end{array}
\rightarrow \begin{array}{c}
|c_1\rangle \\
|c_2\rangle \\
|c_3\rangle \\
|0\rangle \\
|0\rangle \\
U^{c_1c_2c_3}|c_4\rangle \\
\end{array}
\]
Exercise 3 Construction of the Toffoli gate from a control-NOT (Indication : long calculation).

Verify that the control-control-NOT also called Toffoli gate:

\[
\begin{align*}
|c_1\rangle & \rightarrow |c_1\rangle \\
|c_2\rangle & \rightarrow |c_2\rangle \\
|c_2\rangle & \rightarrow |t \oplus c_1 c_2\rangle
\end{align*}
\]

is equivalent to the following circuit made of CNOT, H, T and S:

\[
\begin{align*}
|c_1\rangle & \rightarrow |c_1\rangle \\
|c_2\rangle & \rightarrow |c_2\rangle \\
|t\rangle & \rightarrow |t\rangle \oplus T \oplus T \oplus T \oplus T \oplus T \oplus H \\
& \rightarrow |t \oplus c_1 c_2\rangle
\end{align*}
\]

Exercise 4 Unitary representation of a reversible computation.

Classically a Boolean function \( f \) with inputs \((x_1, \ldots, x_n) \in \{0,1\}^n \) and output in \( \{0,1\} \) can be computed reversibly as

\[
\tilde{f}(x_1, \ldots, x_n; y) = (x_1, \ldots, x_n; y \oplus f(x_1, \ldots, x_n))
\]

where \( y \in \{0,1\} \) is a single storage bit.

This can be implemented in a quantum circuit thanks to the following unitary operation

\[
U_f |x_1, \ldots, x_n; y\rangle = |x_1, \ldots, x_n; y \oplus f(x_1, \ldots, x_n)\rangle
\]

a) What is the Hilbert space relevant for this implementation. Prove that \( U_f \) is indeed a unitary matrix.

b) Generalize this discussion to the case where the output of \( f \) in \( \{0,1\}^m \) (there are \( m \) output bits).