Solution 1.

(a) The effective SNR is

$$\Gamma = \frac{\max\{|H_1|^2, |H_2|^2, \dots, |H_L|^2\}\mathcal{E}_c}{N_0/2}.$$

(b) To find the pdf of Γ, we first find its cumulative distribution function (cdf) and then take its derivative with respect to γ. Starting from the definition, we have ($\overline{\gamma} := 2\mathcal{E}_c/N_0$)

$$\begin{aligned}
\Pr\{\Gamma \le \gamma\} &= \Pr\left\{\max\{|H_1|^2, |H_2|^2, \cdots, |H_L|^2\} \le \gamma/\overline{\gamma}\right\} \\
&= \Pr\left\{\bigcap_{l=1}^{L}\{|H_l|^2 \le \gamma/\overline{\gamma}\}\right\} \\
&= \prod_{l=1}^{L} \Pr\{|H_l|^2 \le \gamma/\overline{\gamma}\} \\
&= (1 - e^{-\gamma/\overline{\gamma}})^L
\end{aligned}$$

and thus

$$f_\Gamma(\gamma) = \begin{cases} \frac{L}{\overline{\gamma}} e^{-\gamma/\overline{\gamma}}(1 - e^{-\gamma/\overline{\gamma}})^{L-1} & \text{if } \gamma \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Following up on (b), the average bit error probability is:

$$\begin{aligned}
\overline{P_\mathrm{e}} &= \int_0^\infty Q(\sqrt{\gamma})\frac{L}{\overline{\gamma}} e^{-\gamma/\overline{\gamma}}(1 - e^{-\gamma/\overline{\gamma}})^{L-1}\, d\gamma \\
&= \int_0^\infty Q(\sqrt{\gamma})\frac{L}{\overline{\gamma}} e^{-\gamma/\overline{\gamma}} \sum_{l=0}^{L-1}\binom{L-1}{l}(-e^{-\gamma/\overline{\gamma}})^l\, d\gamma \\
&= \sum_{l=0}^{L-1}(-1)^l L\binom{L-1}{l}\int_0^\infty Q(\sqrt{\gamma})\frac{1}{\overline{\gamma}}(e^{-\gamma(l+1)/\overline{\gamma}})\, d\gamma \\
&\overset{(1)}{=} \sum_{l=0}^{L-1}(-1)^l\binom{L}{l+1}\int_0^\infty Q(\sqrt{\gamma})\frac{(l+1)}{\overline{\gamma}}(e^{-\gamma(l+1)/\overline{\gamma}})\, d\gamma \\
&\overset{(2)}{=} \sum_{l=0}^{L-1}(-1)^l\binom{L}{l+1}\frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma}}{2(l+1)+\overline{\gamma}}}\right) \\
&= \frac{1}{2}\sum_{l=0}^{L-1}(-1)^l\binom{L}{l+1} + \frac{1}{2}\sum_{l=0}^{L-1}(-1)^{l+1}\binom{L}{l+1}\sqrt{\frac{\overline{\gamma}}{2(l+1)+\overline{\gamma}}} \\
&\overset{(3)}{=} \frac{1}{2} + \frac{1}{2}\sum_{l=1}^{L}(-1)^l\binom{L}{l+1}\sqrt{\frac{\overline{\gamma}}{2l+\overline{\gamma}}} \\
&= \frac{1}{2}\sum_{l=0}^{L}(-1)^l\binom{L}{l+1}\sqrt{\frac{\overline{\gamma}}{2l+\overline{\gamma}}},
\end{aligned}$$

where (1) follows since $L\binom{L-1}{l} = \binom{L}{l+1}(l+1)$ and (2) follows by substituting $\overline{\gamma}$ with $\overline{\gamma}/(l+1)$ in Problem 2(a) of Homework 8 and (3) follows since

$$\sum_{l=0}^{L-1}(-1)^l\binom{L}{l+1} = -\sum_{l=1}^{L}(-1)^l\binom{L}{l}$$
$$= 1-\sum_{l=0}^{L}(-1)^l\binom{L}{l} = 1.$$

(d) When $n=0$, both sides are zero, so the statement is true. Assume that the identity holds when $n=k-1$. Then, we have

$$\begin{aligned}
&\lim_{x\to\infty}\left[x\left(\left(\frac{x}{x-1}\right)^k-1\right)\right]\\
=&\lim_{x\to\infty}\left[\frac{x}{x-1}x\left(\left(\frac{x}{x-1}\right)^{k-1}-1\right)+\frac{x}{x-1}\right]\\
=&\lim_{x\to\infty}\left[\frac{x}{x-1}\right]\lim_{x\to\infty}\left[x\left(\left(\frac{x}{x-1}\right)^{k-1}-1\right)\right]+\lim_{x\to\infty}\left[\frac{x}{x-1}\right]\\
=&1\cdot(k-1)+1=k.
\end{aligned}$$

By induction, Q.E.D.

(e) If we let $x=\overline{\gamma}$ and $n=2l$, then roughly speaking, (d) says that when $\overline{\gamma}$ is very large, we have

$$\begin{aligned}
\overline{\gamma}\left(\left(\frac{\overline{\gamma}}{\overline{\gamma}-1}\right)^{2l}-1\right) &\approx 2l\\
\left(\frac{\overline{\gamma}}{\overline{\gamma}-1}\right)^{2l} &\approx \frac{2l+\overline{\gamma}}{\overline{\gamma}}\\
\left(1-\frac{1}{\overline{\gamma}}\right)^{l} &\approx \sqrt{\frac{\overline{\gamma}}{2l+\overline{\gamma}}}.
\end{aligned}$$

Therefore, at high SNR, the error probability can be approximated as

$$\begin{aligned}
\overline{P_e} &\approx \frac{1}{2}\sum_{l=0}^{L}(-1)^l\binom{L}{l}\left(1-\frac{1}{\overline{\gamma}}\right)^l\\
&= \sum_{l=0}^{L}\binom{L}{l}\left(\frac{1}{\overline{\gamma}}-1\right)^l 1^{L-l}\\
&= \frac{1}{2}\left(\left(\frac{1}{\overline{\gamma}}-1\right)+1\right)^L\\
&= \frac{1}{2\overline{\gamma}^L},
\end{aligned}$$

which says that the diversity order of selection combining is L.

SOLUTION 2.

(a) If we send the same information symbols u over two transmit antennas, the received signal at each antenna $j, j = 1, \ldots, R$ is

$$Y_j[0] = (H_{j,1} + H_{j,2})u + Z_j[0]$$

Since $H_{j,1}$ and $H_{j,2}$ are i.i.d complex Gaussian with unit variance, the random variable $H_{j,1} + H_{j,2}$ is also complex Gaussian with variance 2. So this is equivalent to a fading channel

$$Y_j[0] = \tilde{H}_j u + Z_j[0]$$

with $\tilde{H}_j := H_{j,1} + H_{j,2}$. With R such received signals, we can obviously only obtain a diversity of R.

(b) Now suppose we can use the channel for time $n = 0, 1$. If we set $x_1[0] = u, x_2[0] = 0$ and $x_1[1] = 0, x_2[1] = u$, this is equivalent to $2R$ independent channels as

$$\begin{aligned} Y_j[0] &= H_{j,1}u + Z_j[0] \quad \text{for } j = 1, \ldots, R \\ Y_j[1] &= H_{j,2}u + Z_j[1] \quad \text{for } j = 1, \ldots, R \end{aligned}$$

From Lecture Notes, section 5.4 we know this can provide us a diversity of $2R$ for the information u.

(c) For convenience, we rewrite the input-output relation in vector form:

$$\boldsymbol{y}[n] = \boldsymbol{h}_1 x_1[n] + \boldsymbol{h}_2 x_2[n] + \boldsymbol{z}[n],$$

where $\boldsymbol{h}_k = \begin{pmatrix} H_{1k} & H_{2k} & \cdots & H_{Rk} \end{pmatrix}^T$, $k \in \{1, 2\}$. Using the Alamouti scheme, we have

$$\begin{aligned} \boldsymbol{y}[0] &= \boldsymbol{h}_1 u_1 + \boldsymbol{h}_2 u_2 + \boldsymbol{z}[0] \\ \boldsymbol{y}[1] &= \boldsymbol{h}_1(-u_2^*) + \boldsymbol{h}_2 u_1^* + \boldsymbol{z}[1]. \end{aligned}$$

Now we can process the received signals as

$$\begin{aligned} \tilde{\boldsymbol{y}}[0] &= \boldsymbol{h}_1^* \boldsymbol{y}[0] + \boldsymbol{y}^*[1]\boldsymbol{h}_2 = (\|\boldsymbol{h}_1\|^2 + \|\boldsymbol{h}_2\|^2)u_1 + (\boldsymbol{h}_1^* \boldsymbol{z}[0] + \boldsymbol{z}^*[1]\boldsymbol{h}_2) \\ \tilde{\boldsymbol{y}}[1] &= \boldsymbol{h}_2^* \boldsymbol{y}[0] - \boldsymbol{y}^*[1]\boldsymbol{h}_1 = (\|\boldsymbol{h}_1\|^2 + \|\boldsymbol{h}_2\|^2)u_2 + (\boldsymbol{h}_2^* \boldsymbol{z}[0] - \boldsymbol{z}^*[1]\boldsymbol{h}_1). \end{aligned}$$

Clearly, in two channel uses, we attain diversity $2R$ for both information symbols.

SOLUTION 3.

(a) This is similar to what we did in the Homework 7 (when we used OFDM for broadcasting). Clearly, the cyclic prefix has to be at least 3 (since the earliest sample we can see has time index $n - 3$).

We first rewrite the channel model as

$$
\begin{pmatrix} y_1[0] \\ y_1[1] \\ y_1[2] \\ \vdots \\ y_1[N-3] \\ y_1[N-2] \\ y_1[N-1] \end{pmatrix}
=
\begin{pmatrix}
1 & 0 & \dots & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 1 & 0 & \dots & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 1 & \dots & 0 & 0 & \frac{1}{2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & 0 & \frac{1}{2} & 1 & 0 & 0 \\
0 & \dots & \frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 \\
0 & \dots & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 1
\end{pmatrix}
\begin{pmatrix} x_1[0] \\ x_1[1] \\ x_1[2] \\ \vdots \\ x_1[N-3] \\ x_1[N-2] \\ x_1[N-1] \end{pmatrix}
$$

$$
+
\begin{pmatrix}
0 & 0 & \dots & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\
-\frac{1}{2} & 0 & 0 & \dots & 0 & \frac{1}{2} & 0 \\
0 & -\frac{1}{2} & 0 & \dots & 0 & 0 & -\frac{1}{2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & 0 & -\frac{1}{2} & 0 & 0 & 0 \\
0 & \dots & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\
0 & \dots & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0
\end{pmatrix}
\begin{pmatrix} x_2[0] \\ x_2[1] \\ x_2[2] \\ \vdots \\ x_2[N-3] \\ x_2[N-2] \\ x_2[N-1] \end{pmatrix}
+
\begin{pmatrix} w_1[0] \\ w_1[1] \\ w_1[2] \\ \vdots \\ w_1[N-3] \\ w_1[N-2] \\ w_1[N-1] \end{pmatrix}
\quad (1)
$$

and

$$
\begin{pmatrix} y_2[0] \\ y_2[1] \\ y_2[2] \\ \vdots \\ y_2[N-3] \\ y_2[N-2] \\ y_2[N-1] \end{pmatrix}
=
\begin{pmatrix}
0 & 0 & \dots & 0 & 0 & \frac{1}{6} & \frac{1}{3} \\
\frac{1}{3} & 0 & 0 & \dots & 0 & 0 & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{3} & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 \\
0 & \dots & 0 & \frac{1}{6} & \frac{1}{3} & 0 & 0 \\
0 & \dots & 0 & 0 & \frac{1}{6} & \frac{1}{3} & 0
\end{pmatrix}
\begin{pmatrix} x_1[0] \\ x_1[1] \\ x_1[2] \\ \vdots \\ x_1[N-3] \\ x_1[N-2] \\ x_1[N-1] \end{pmatrix}
$$

$$
+
\begin{pmatrix}
1 & 0 & \dots & 0 & 0 & -\frac{1}{2} & 0 \\
0 & 1 & 0 & \dots & 0 & 0 & -\frac{1}{2} \\
-\frac{1}{2} & 0 & 1 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & -\frac{1}{2} & 0 & 1 & 0 & 0 \\
0 & \dots & 0 & -\frac{1}{2} & 0 & 1 & 0 \\
0 & \dots & 0 & 0 & -\frac{1}{2} & 0 & 1
\end{pmatrix}
\begin{pmatrix} x_2[0] \\ x_2[1] \\ x_2[2] \\ \vdots \\ x_2[N-3] \\ x_2[N-2] \\ x_2[N-1] \end{pmatrix}
+
\begin{pmatrix} w_2[0] \\ w_2[1] \\ w_2[2] \\ \vdots \\ w_2[N-3] \\ w_2[N-2] \\ w_2[N-1] \end{pmatrix}
\quad (2)
$$

We will use $\boldsymbol{x}_1, \boldsymbol{x}_2$ to denote the vectors $(x_1[0], \ldots, x_1[N-1])$ and $(x_2[0], \ldots, x_2[N-1])$

and so on. Rewrite the (1) as

$$F_N \boldsymbol{y}_1 = F_N \begin{pmatrix} 1 & 0 & \dots & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 & \dots & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 & \dots & 0 & 0 & \frac{1}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \frac{1}{2} & 1 & 0 & 0 \\ 0 & \dots & \frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 \\ 0 & \dots & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 1 \end{pmatrix} F_N^{-1} \boldsymbol{X}_1$$

$$+ F_N \begin{pmatrix} 0 & 0 & \dots & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \dots & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & \dots & 0 & 0 & -\frac{1}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \dots & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \dots & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{pmatrix} F_N^{-1} \boldsymbol{X}_2 + F_N \begin{pmatrix} w_1[0] \\ w_1[1] \\ w_1[2] \\ \vdots \\ w_1[N-3] \\ w_1[N-2] \\ w_1[N-1] \end{pmatrix}$$

where F_N is the Fourier matrix of length N and $\boldsymbol{X}_j = F_N \boldsymbol{x}_j, j = 1, 2$. Since all circulant matrices can be diagonalized by the Fourier matrix, the above expression will take the form

$$\boldsymbol{Y}_1 = H_{11} \boldsymbol{X}_1 + H_{12} \boldsymbol{X}_2 + \boldsymbol{W}_1$$

where H_{11}, H_{12} are two diagonal matrices. Using the same argument to (2) gives the OFDM system

$$\boldsymbol{Y}_2 = H_{21} \boldsymbol{X}_1 + H_{22} \boldsymbol{X}_2 + \boldsymbol{W}_2$$

where the diagonal matrices H_{21}, H_{22} are obtained using F_N and channel matrices in (2).

(b) In the case $N = 4$, the four matrices are give by

$$H_{11} = \text{diag}(2, 1, 0, 1) \qquad H_{12} = \text{diag}(-1, 0, 1, 0),$$
$$H_{21} = \text{diag}(\tfrac{1}{2}, -\tfrac{1}{6} + j\tfrac{1}{3}, -\tfrac{1}{6} - j\tfrac{1}{3}, -\tfrac{1}{6}), \qquad H_{22} = \text{diag}(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{3}{2})$$

(c) If we view $\boldsymbol{X}_1, \boldsymbol{X}_2$ as channel inputs and $\boldsymbol{Y}_1, \boldsymbol{Y}_2$ as channel outputs, the whole system can be thought as an 8×8 MIMO system of the form

$$\begin{pmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{21} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$$

For the case $N = 4$ and the block matrix $\begin{pmatrix} H_{11} & H_{21} \\ H_{21} & H_{22} \end{pmatrix}$ can be diagonalized using SVD, and the resulting 8 parallel channels have channel gains

$$\Sigma = \text{diag}(2.25, 1.58, 1.58, 1.12, 0.95, 0.95, 0.67, 0.15).$$

Solution 4.

(a) With the choice of $N = 2$, it is easy to find out the expressions of OFDM channel coefficients:

$$\begin{aligned} H_{11} &= h_{10} + h_{11} \\ G_{12} &= g_{20} + g_{21} \\ G_{21} &= g_{10} + g_{11} \\ H_{22} &= h_{20} + h_{21} \end{aligned}$$

Since each OFDM channel coefficients is a sum of two independent (and different) circularly symmetric complex Gaussian random variables , they are also circularly symmetric complex Gaussian and independent. Furthermore as h_{ij}, g_{ij} are independent and have variance 1, each H_{ij}, G_{ij} should have variance 2.

(b) Applying the zero-forcing filter on the channel output gives the following expression

$$\begin{aligned} \begin{pmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{pmatrix} &:= \begin{pmatrix} H_{11} & G_{12} \\ G_{21} & H_{22} \end{pmatrix}^{-1} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \\ &= \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} H_{11} & G_{12} \\ G_{21} & H_{22} \end{pmatrix}^{-1} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \\ &= \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \frac{1}{H_{11}H_{22} - G_{21}G_{12}} \begin{pmatrix} H_{22} & -G_{12} \\ -G_{21} & H_{11} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \end{aligned}$$

To detect X_1, and because only real part of X_1 carries information, we are only interested in the quantity

$$\begin{aligned} \Re\{\tilde{Y}_1\} &= \Re\{X_1\} + \Re\{\tilde{Z}_1\} \\ &= X_1 + \Re\{\tilde{Z}_1\} \end{aligned}$$

where $\tilde{Z}_1 := \frac{H_{22}Z_1 - G_{12}Z_2}{H_{11}H_{22} - G_{21}G_{12}}$. Conditioned on $H_{11}, H_{11}, G_{12}, G_{21}$, the variance of $\Re\{\tilde{Z}_1\}$ is

$$\tilde{N} := \text{var}(\Re\{\tilde{Z}_1\} | H_{11}, H_{22}, G_{12}, G_{21}) = \frac{(|H_{22}|^2 + |G_{12}|^2)N_0/2}{|H_{11}H_{22} - G_{21}G_{12}|^2}.$$

and the conditional error probability is

$$P_{\text{e}|H,G} = Q\left(\frac{d}{2\sqrt{\tilde{N}}}\right) = Q\left(\sqrt{\frac{\mathcal{E}}{\tilde{N}}}\right)$$

Finally the average error probability (averaged over the channel coefficients) can be written as the integral

$$P_e = \int_{H_{11}} \int_{H_{22}} \int_{G_{12}} \int_{G_{21}} P_{e|H,G} \frac{1}{2\pi} e^{-\frac{|H_{11}|^2}{2}} \frac{1}{2\pi} e^{-\frac{|H_{22}|^2}{2}} \frac{1}{2\pi} e^{-\frac{|G_{12}|^2}{2}} \frac{1}{2\pi} e^{-\frac{|G_{21}|^2}{2}} dH_{11} dH_{22} dG_{12} dG_{21}$$

as we have shown in (a) that all channel coefficients are independent circularly symmetric complex Gaussian with variance 2.

Solution 5.

(a) Since H is circularly symmetric complex Gaussian with unit variance, the received signal $(Y[0], Y[1])$ is distributed as $(\mathcal{CN}(0, \mathcal{E}_c + N_0), \mathcal{CN}(0, N_0))$ if $\boldsymbol{x} = (\sqrt{\mathcal{E}_c}, 0)$ is sent, and distributed as $(\mathcal{CN}(0, N_0), \mathcal{CN}(0, \mathcal{E}_c + N_0))$ if $\boldsymbol{x} = (0, \sqrt{\mathcal{E}_c})$ is sent. Using the fact that $Y[0]$ and $Y[1]$ are independent, the conditional probability of $\boldsymbol{Y} = (Y[0], Y[1])$ given $\boldsymbol{x} = (x[0], x[1])$ is

$$
\begin{aligned}
p_{\boldsymbol{Y}|\boldsymbol{X}}\left(\boldsymbol{y}\middle|\boldsymbol{x} = \begin{pmatrix} \sqrt{\mathcal{E}_c} \\ 0 \end{pmatrix}\right) &= \frac{1}{\pi^2 N_0(\mathcal{E}_c + N_0)} e^{-\frac{|y[0]|^2}{\mathcal{E}_c + N_0}} \cdot e^{-\frac{|y[1]|^2}{N_0}} \\
p_{\boldsymbol{Y}|\boldsymbol{X}}\left(\boldsymbol{y}\middle|\boldsymbol{x} = \begin{pmatrix} 0 \\ \sqrt{\mathcal{E}_c} \end{pmatrix}\right) &= \frac{1}{\pi^2 N_0(\mathcal{E}_c + N_0)} e^{-\frac{|y[1]|^2}{\mathcal{E}_c + N_0}} \cdot e^{-\frac{|y[0]|^2}{N_0}}
\end{aligned}
$$

(b) To get a cleaner expression, we write out the *log-likelihood* of the two conditional probabilities:

$$
\begin{aligned}
\Lambda_{01}(\boldsymbol{y}) &:= \ln \left(\frac{p_{\boldsymbol{Y}|\boldsymbol{X}}\left(\boldsymbol{y}\middle|\boldsymbol{x} = \begin{pmatrix} \sqrt{\mathcal{E}_c} \\ 0 \end{pmatrix}\right)}{p_{\boldsymbol{Y}|\boldsymbol{X}}\left(\boldsymbol{y}\middle|\boldsymbol{x} = \begin{pmatrix} 0 \\ \sqrt{\mathcal{E}_c} \end{pmatrix}\right)} \right) \\
&= \frac{(|y[0]|^2 - |y[1]|^2)\mathcal{E}_c}{(\mathcal{E}_c + N_0)N_0}
\end{aligned}
$$

In terms of $\Lambda_{01}(\boldsymbol{y})$, the ML detection is that we decide for the first message if $\Lambda_{01}(\boldsymbol{y}) \geq 0$ and for the second message otherwise, which in turn is just to compare $|y[0]|^2$ and $|y[1]|^2$.

(c) To calculate the error probability, notice that in this case $|y[0]|^2$ and $|y[1]|^2$ are both exponentially distributed with mean $\mathcal{E} + N_0$ and N_0, respectively. The error probability can be calculated as follows.

$$
\begin{aligned}
\overline{P_e} &\overset{(1)}{=} \Pr\left\{ |y[0]|^2 < |y[1]|^2 \middle| \boldsymbol{x} = \begin{pmatrix} \sqrt{\mathcal{E}_c} \\ 0 \end{pmatrix} \right\} \\
&= \int_0^\infty \frac{1}{N_0} e^{-\frac{v_1}{N_0}} \int_0^{v_1} \frac{1}{\mathcal{E} + N_0} e^{-\frac{v_0}{\mathcal{E} + N_0}} \, dv_0 \, dv_1 \\
&= \frac{N_0}{2N_0 + \mathcal{E}} \\
&= \frac{2}{4 + \bar{\gamma}},
\end{aligned}
$$

where (1) follows by symmetry and $\bar{\gamma} := \frac{\mathcal{E}_c}{N_0/2}$. We see that the error probability decays inversely proportional to $\bar{\gamma}$.

(d) Diversity order is 1.

(e) Perform hard decision on every channel separately and then use majority decoding. As we saw in Homework 8, Problem 3, the probability of error of this scheme is upper bounded by $\alpha p^{\lceil L/2 \rceil}$ where p is the probability of error for a single channel use. Since we have a diversity of 1 for one channel use, we will obtain a diversity order of $\lceil L/2 \rceil$ with L channel use.

(f) First we derive the ML detector:

$$
\begin{aligned}
\Lambda_{01}(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_L) &= \ln \frac{\Pr\left\{\boldsymbol{y}_1 = \begin{pmatrix} y_1[0] \\ y_1[1] \end{pmatrix},\ldots,\boldsymbol{y}_L = \begin{pmatrix} y_L[0] \\ y_L[1] \end{pmatrix} \middle| \boldsymbol{x} = \begin{pmatrix} \sqrt{\mathcal{E}_c} \\ 0 \end{pmatrix}\right\}}{\Pr\left\{\boldsymbol{y}_1 = \begin{pmatrix} y_1[0] \\ y_1[1] \end{pmatrix},\ldots,\boldsymbol{y}_L = \begin{pmatrix} y_L[0] \\ y_L[1] \end{pmatrix} \middle| \boldsymbol{x} = \begin{pmatrix} 0 \\ \sqrt{\mathcal{E}_c} \end{pmatrix}\right\}} \\
&= \ln \frac{e^{-\frac{|y_1[0]|^2+\cdots+|y_L[0]|^2}{\mathcal{E}_c+N_0}} e^{-\frac{|y_1[1]|^2+\cdots+|y_L[1]|^2}{N_0}}}{e^{-\frac{|y_1[0]|^2+\cdots+|y_L[0]|^2}{N_0}} e^{-\frac{|y_1[1]|^2+\cdots+|y_L[1]|^2}{\mathcal{E}_c+N_0}}} \\
&= \frac{(|y_1[0]|^2+\cdots+|y_L[0]|^2-|y_1[1]|^2-\cdots-|y_L[1]|^2)\mathcal{E}_c}{N_0(\mathcal{E}_c+N_0)}.
\end{aligned}
$$

For the sake of convenience, define

$$V[i] = |Y_1[i]|^2 + \cdots + |Y_L[i]|^2, \text{ for } i = 0, 1.$$

At this point it is clear that the ML decoder should decode $\begin{pmatrix} \sqrt{\mathcal{E}_c} \\ 0 \end{pmatrix}$ if $V[0] \geq V[1]$ and decode $\begin{pmatrix} 0 \\ \sqrt{\mathcal{E}_c} \end{pmatrix}$ otherwise.

By symmetry, the probability of error is

$$P_{\mathrm{e}} = \Pr\left\{V[0] \leq V[1] \middle| \boldsymbol{x} = \begin{pmatrix} \sqrt{\mathcal{E}_c} \\ 0 \end{pmatrix}\right\}.$$

Note that conditioned on transmission of $(\sqrt{\mathcal{E}_c}, 0)^T$, the random variable $V[0]$ is the sum of squares of $2L$ independent Gaussians with variance $\mathcal{E}_c/2 + N_0/2$ and $V[1]$ is the sum of squares of $2L$ independent Gaussians with variance $N_0/2$. We can respectively define $U[0] = \frac{V[0]}{\mathcal{E}_c/2+N_0/2}$ and $U[1] = \frac{V[1]}{N_0/2}$. Following the hint, $U[0]$ and $U[1]$ are independent χ-squared random variables of degree $2L$.

$$P_{\mathrm{e}} = \Pr\left\{(\mathcal{E}_c/2 + N_0/2)U[0] \leq (N_0/2)U[1]\right\} = \Pr\left\{\frac{U[0]}{U[1]} \leq \frac{N_0/2}{\mathcal{E}_c/2 + N_0/2}\right\}.$$

Defining $W = \frac{U[0]}{U[1]}$ proves the claim, with $\alpha = \frac{N_0/2}{\mathcal{E}_c/2+N_0/2} = \frac{2}{2+\bar{\gamma}}$.

(g) Based on the hint, the probability of error is given by:

$$P_{\mathrm{e}} = I_{\frac{\alpha}{\alpha+1}}(L, L). \tag{3}$$

Using the recursive equation we obtain:

$$
\begin{aligned}
P_{\mathrm{e}} &= \frac{(\frac{\alpha}{\alpha+1})^L(\frac{1}{\alpha+1})^{L-1}}{(L-1)B(L,L-1)} + I_{\frac{\alpha}{\alpha+1}}(L, L-1) \\
&= \frac{(\frac{\alpha}{\alpha+1})^L(\frac{1}{\alpha+1})^{L-1}}{(L-1)B(L,L-1)} + \frac{(\frac{\alpha}{\alpha+1})^L(\frac{1}{\alpha+1})^{L-2}}{(L-2)B(L,L-2)} + I_{\frac{\alpha}{\alpha+1}}(L, L-2) \\
&= \ldots \\
&= \frac{(\frac{\alpha}{\alpha+1})^L(\frac{1}{\alpha+1})^{L-1}}{(L-1)B(L,L-1)} + \cdots + \frac{(\frac{\alpha}{\alpha+1})^L(\frac{1}{\alpha+1})^{1}}{B(L,1)} + (\frac{\alpha}{\alpha+1})^L.
\end{aligned}
$$

(h) Since $\alpha > 0$, we have $\frac{1}{\alpha+1} < 1$. Therefore,

$$P_{\mathrm{e}} \leq (\frac{\alpha}{\alpha+1})^L \left[\frac{1}{(L-1)B(L,L-1)} + \cdots + \frac{1}{B(L,1)} + 1\right]. \tag{4}$$

Finally, we have

$$\frac{\alpha}{\alpha+1} = \frac{\frac{2}{2+\bar{\gamma}}}{1+\frac{2}{2+\bar{\gamma}}} = \frac{1}{1+\bar{\gamma}/2} \leq 2/\bar{\gamma}. \tag{5}$$

Defining $\theta = 2^L \left[\frac{1}{(L-1)B(L,L-1)} + \cdots + \frac{1}{B(L,1)} + 1\right]$, we can write:

$$P_{\mathrm{e}} \leq \theta \frac{1}{\bar{\gamma}^L}. \tag{6}$$

Therefore, the diversity of the ML detector is L.