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Handout 17 Advanced Digital Communications
Solutions to Homework 7 Nov. 7, 2016

SOLUTION 1.

(a) The easiest approach is to consider the power spectral density of the equivalent noise $\tilde{V}[n]$, which is simply the original noise $V[n]$ passed through the filter $1/D(z)$.

$$\begin{aligned} S_{\tilde{V}}(z) &= \frac{1}{D(z)}\frac{1}{D^*(1/z^*)}S_V(z) = \frac{1}{D(z)}\frac{1}{D^*(1/z^*)}\frac{N_0}{2}D(z) \\ &= \frac{N_0}{2}\frac{1}{D^*(1/z^*)} = \frac{N_0}{2}\left((1-\alpha(1/z^*))(1-\alpha z^*)\right)^* \\ &= \frac{N_0}{2}(1-\alpha z^{-1})(1-\alpha z) \\ &= \frac{N_0}{2}\left((1+\alpha^2)-\alpha z-\alpha z^{-1}\right). \end{aligned}$$

The noise variance we are looking for satisfies

$$\mathbb{E}[\tilde{V}[n]^2] = R_{\tilde{V}}[k=0],$$

where $R_{\tilde{V}}[k]$ is the autocorrelation function, which is simply the inverse Z-transform of the power spectral density $S_{\tilde{V}}(z)$. But we can read this directly out of the formula:

$$\mathbb{E}[\tilde{V}[n]^2] = R_{\tilde{V}}[k=0] = \frac{N_0}{2}(1+\alpha^2).$$

(b)

$$P_e^{(ZF)} = Q\left(\frac{d}{2\sqrt{\sigma^2}}\right),$$

where $d = 2\sqrt{\mathcal{E}}$ is the distance between the two message points, and $\sigma^2 = \frac{N_0}{2}(1+\alpha^2)$ is the variance of the Gaussian noise that affects our decision, hence,

$$P_e^{(ZF)} = Q\left(\sqrt{\frac{2\mathcal{E}}{N_0(1+\alpha^2)}}\right),$$

(c) This one was conceptually somewhat new. Clearly, we start by plugging in what we know about $I[n]$ to obtain

$$S[n] = \begin{cases} \alpha^n I[0] + W[n], & \text{for } n \geq 0, \\ W[n], & \text{for } n < 0. \end{cases}$$

Since the noise $W[n]$ is AWGN, the samples $S[n]$ for $n < 0$ are pure noise and thus, irrelevant information. Moreover, the noiseless transmitted sequence $\alpha^n I[0]$ (for $n \geq 0$) can only assume two different values:

$$\begin{aligned} \text{If } I[0] = \sqrt{\mathcal{E}}: &\quad \sqrt{\mathcal{E}} \quad (1, \alpha, \alpha^2, \alpha^3, \alpha^4, \ldots) \\ \text{If } I[0] = -\sqrt{\mathcal{E}}: &\quad -\sqrt{\mathcal{E}} \quad (1, \alpha, \alpha^2, \alpha^3, \alpha^4, \ldots). \end{aligned}$$

Hence, all we are doing is to distinguish between two possible sequences in AWGN of power $N_0/2$, which is the basic problem discussed in Chapter 3. Since we know from class that under AWGN, the ML detector is minimum distance, the error probability is simply given by

$$P_e = Q\left(\frac{d}{2\sqrt{N_0/2}}\right),$$

where d is the distance between the two possible message points:

$$d^2 = \mathcal{E}(2^2 + (2\alpha)^2 + (2\alpha^2)^2 + (2\alpha^3)^2 + (2\alpha^4)^2 + \ldots) = 4\mathcal{E}\sum_{k=0}^{\infty}\alpha^{2k} = 4\mathcal{E}\frac{1}{1-\alpha^2}.$$

Hence, we find as our desired lower bound

$$P_e^{(lowerbound)} = Q\left(\sqrt{\frac{2\mathcal{E}}{N_0(1-\alpha^2)}}\right).$$

(d) Let us denote the energy used by the ZF as $\mathcal{E}_{ZF}$. We have to equate $P_e^{(ZF)}$ to the lower bound:

$$Q\left(\sqrt{\frac{2\mathcal{E}_{ZF}}{N_0(1+\alpha^2)}}\right) = Q\left(\sqrt{\frac{2\mathcal{E}}{N_0(1-\alpha^2)}}\right).$$

The two are equal if and only if the arguments inside the Q-function are equal, meaning that

$$\frac{2\mathcal{E}_{ZF}}{N_0(1+\alpha^2)} = \frac{2\mathcal{E}}{N_0(1-\alpha^2)}.$$

That is, we need

$$\mathcal{E}_{ZF} = \mathcal{E}\frac{1+\alpha^2}{1-\alpha^2}.$$

What we can see is that when $\alpha = 0$, Zero-forcing is optimal (not surprisingly), but as α tends to one, the power penalty of Zero-forcing tends to infinity.

Perhaps a better question would be: If I allow you to use twice the power of the lower bound (the usual 3dB), how large an α can you tolerate? That we, we must have

$$\frac{1+\alpha^2}{1-\alpha^2} \leq 2,$$

which, as you can verify, says that we can tolerate up to $|\alpha| \leq \frac{1}{\sqrt{3}} \approx 0.577$, showing that zero-forcing is not that bad after all.

Solution 2.

(a) For user A, the minimum cyclic prefix is 3, and for user B it is 2. Hence, if you use a cyclic prefix of length 3, both users will have no ISI.

(b)

$$H_m = \sum_{\ell=0}^{N-1} h_\ell \omega^{\ell m}$$

where

$$\omega = e^{-j2\pi/N} = -j \text{ for the case } N = 4.$$

For the channel to user B, we have $h_0 = \sqrt{\frac{3}{2}}, h_1 = 0, h_2 = \sqrt{\frac{3}{2}}$, and $h_3 = 0$. That is, we find

$$\begin{aligned}
H_0 &= h_0 + h_1 + h_2 + h_3 = \sqrt{6} \\
H_1 &= h_0 + (-j)h_1 + (-j)^2 h_2 + (-j)^3 h_3 = h_0 - jh_1 - h_2 + jh_3 = 0 \\
H_2 &= h_0 + (-j)^2 h_1 + (-j)^4 h_2 + (-j)^6 h_3 = h_0 - h_1 + h_2 - h_3 = \sqrt{6} \\
H_3 &= h_0 + (-j)^3 h_1 + (-j)^6 h_2 + (-j)^9 h_3 = h_0 + jh_1 - h_2 - jh_3 = 0.
\end{aligned}$$

(c) Using the same reasoning as in (b) we can see that the channel coefficients for user A will be

$$H_0 = 2, \quad H_1 = 1, \quad , H_2 = 0, \quad H_3 = 1.$$

There are, hence, only two possible transmitted message points of length $N = 4$:

$$\begin{aligned}
\boldsymbol{x}_1 &= \sqrt{\mathcal{E}}(2, 1, 0, 1) \\
\boldsymbol{x}_2 &= -\sqrt{\mathcal{E}}(2, 1, 0, 1)
\end{aligned}$$

The only slight difficulty was that instead of transmitting over the real-valued Gaussian vector channel of length $N = 4$ (like in chapter 3), we are now using the complex-valued Gaussian vector channel.

The key observation was to see that the imaginary parts of the 4 received signals contain only noise (since there is no signal component in the imaginary parts). Moreover, since real and imaginary parts of the circularly symmetric complex-valued Gaussian noise are independent of each other, the imaginary parts of the received symbols are irrelevant information, and their real parts are a sufficient statistic.

Hence, since we know from class that under AWGN, the ML detector is minimum distance, the ML detector is given by first keeping the real parts of the received symbols only, and then by finding the closer of the two possible message points $\boldsymbol{x}_1$ or $\boldsymbol{x}_2$.

Finding the corresponding error probability is again the same old formula:

$$P_e = Q\left(\frac{d}{2\sqrt{N_0/2}}\right),$$

and the distance between our message points is

$$d^2 = \mathcal{E}(4^2 + 2^2 + 2^2) = 24\mathcal{E},$$

thus,

$$P_e = Q\left(\sqrt{\frac{12\mathcal{E}}{N_0}}\right).$$

(d) Clearly, we give Channel 2 to user B, since it is useless for user A. Moreover, Channels 1 and 3 should probably go to user A. The only tricky one is Channel 0, since it is good for both users. Let us start by giving Channel 0 to user A. Clearly, since Channel 2 is anyway useless for user A, the error probability for user A will be exactly as in (c), namely

$$P_e^{\text{user A}} = Q\left(\sqrt{\frac{12\mathcal{E}}{N_0}}\right).$$

In this case, user B only has channel 2, which is really easy to analyze: we have a distance of $d = 2\sqrt{6\mathcal{E}}$, and hence,

$$P_e^{\text{user B}} = Q\left(\sqrt{\frac{12\mathcal{E}}{N_0}}\right).$$

Both error probabilities are equal in this case.

Now, clearly, if we give anything more to user B, then user B will get better, and user A will get worse. So, this is the fairest solution that simultaneously minimizes error probabilities.

(e) This one was the most difficult question of this problem, and you had to think a little bit more freely. There are many ways to obtain the answer. One is to go back to part (c) and redo this allowing to spread the total energy $4\mathcal{E}$ arbitrarily among the four channels. Let us say that we place $\alpha_0^2\mathcal{E}$ in channel 0, $\alpha_1^2\mathcal{E}$ in channel 1, and so on, with $\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 4$. Then, following the analysis in part (c), we see that we are distinguishing between the following two possible signal points:

$$\begin{aligned} \boldsymbol{x}_1 &= \sqrt{\mathcal{E}}(2\alpha_0, \alpha_1, 0, \alpha_3) \\ \boldsymbol{x}_2 &= -\sqrt{\mathcal{E}}(2\alpha_0, \alpha_1, 0, \alpha_3) \end{aligned}$$

and thus, the distance is

$$d^2 = \mathcal{E}\left((4\alpha_0)^2 + (2\alpha_1)^2 + (2\alpha_3)^2\right) = \mathcal{E}\left(16\alpha_0^2 + 4\alpha_1^2 + 4\alpha_3^2\right).$$

As always, the larger d^2, the smaller an error probability we attain. But you can then directly see that you should select $\alpha_0^2 = 4$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0$, that is, you should **only** use the best channel! A second observation from this derivation is that if you have two channels of equal quality, you can assign powers between them any way you want — that is, if there are two best channels, you can use one, the other, or both, without changing the performance at all.

But from these insights, you can immediately conclude that the best is for user A to use Channel 0, and for user B to use Channel 2. Channels 1 and 3 are left unused.

Finally, for the energy: Let us denote the energy for user A by $\mathcal{E}_A$ and the energy for user B by $\mathcal{E}_B$. Then, we have

$$\begin{aligned} P_e^{\text{user A}} &= Q\left(\sqrt{\frac{8\mathcal{E}_A}{N_0}}\right) \\ P_e^{\text{user B}} &= Q\left(\sqrt{\frac{12\mathcal{E}_B}{N_0}}\right). \end{aligned}$$

To make both error probabilities the same, we need to have $8\mathcal{E}_A = 12\mathcal{E}_B$. Moreover, we need to satisfy $\mathcal{E}_A + \mathcal{E}_B = 4$. So, two variables, two equations — we easily find the allocation:

$$\mathcal{E}_A = \frac{12}{5}, \qquad \mathcal{E}_B = \frac{8}{5}.$$

SOLUTION 3.

(a) α is the probability that the noise is very negative, so that although the transmitted signal is $\sqrt{\mathcal{E}}$, the noise Z pushes the received signal $Y = x + Z$ below the threshold $-\theta\sqrt{\mathcal{E}}$. Formally,

$$\alpha = \Pr\{D = 0 | x = \sqrt{\mathcal{E}}\} = \Pr\{Z < -(1+\theta)\sqrt{\mathcal{E}}\} = Q\left(\frac{(1+\theta)\sqrt{\mathcal{E}}}{\sqrt{N_0/2}}\right).$$

To compute β, by analogy to α, we can easily write:

$$\begin{aligned}\beta = \Pr\{D = * | x = \sqrt{\mathcal{E}}\} &= \Pr\{-(1+\theta)\sqrt{\mathcal{E}} \le Z < -(1-\theta)\sqrt{\mathcal{E}}\} \\ &= Q\left(\frac{(1-\theta)\sqrt{\mathcal{E}}}{\sqrt{N_0/2}}\right) - Q\left(\frac{(1+\theta)\sqrt{\mathcal{E}}}{\sqrt{N_0/2}}\right).\end{aligned}$$

(b) You can express the likelihoods conveniently by letting $k(\boldsymbol{d})$ denote the number of ones in $\boldsymbol{d}$, $\ell(\boldsymbol{d})$ denote the number of zeroes in $\boldsymbol{d}$, and $s(\boldsymbol{d})$ denote the number of stars in $\boldsymbol{d}$. Then, you can write for any given sequence $\boldsymbol{d}$

$$\begin{aligned}\Pr\{\boldsymbol{D} | \sqrt{\mathcal{E}} \text{ transmitted}\} &= (1-\alpha-\beta)^{k(\boldsymbol{d})}\alpha^{\ell(\boldsymbol{d})}\beta^{s(\boldsymbol{d})} \\ \Pr\{\boldsymbol{D} = \boldsymbol{d} | -\sqrt{\mathcal{E}} \text{ transmitted}\} &= (1-\alpha-\beta)^{\ell(\boldsymbol{d})}\alpha^{k(\boldsymbol{d})}\beta^{s(\boldsymbol{d})}\end{aligned}$$

The ML can then be found easily from the Likelihood Ratio:

$$\begin{aligned}\Lambda_{12} &= \frac{\Pr\{\boldsymbol{D} = \boldsymbol{d} | \sqrt{\mathcal{E}} \text{ transmitted}\}}{\Pr\{\boldsymbol{D} = \boldsymbol{d} | -\sqrt{\mathcal{E}} \text{ transmitted}\}} \\ &= \frac{(1-\alpha-\beta)^{k(\boldsymbol{d})}\alpha^{\ell(\boldsymbol{d})}\beta^{s(\boldsymbol{d})}}{(1-\alpha-\beta)^{\ell(\boldsymbol{d})}\alpha^{k(\boldsymbol{d})}\beta^{s(\boldsymbol{d})}} \\ &= \frac{(1-\alpha-\beta)^{k(\boldsymbol{d})}\alpha^{\ell(\boldsymbol{d})}}{(1-\alpha-\beta)^{\ell(\boldsymbol{d})}\alpha^{k(\boldsymbol{d})}}.\end{aligned}$$

In other words, a sufficient statistic for $\boldsymbol{d}$ is the number of ones and the number of zeroes in the vector — the number of $*$ is irrelevant information and can be dropped. If there are more ones than zeroes, the ML says that $\sqrt{\mathcal{E}}$ was transmitted; if there are more zeroes than ones, the ML says that $-\sqrt{\mathcal{E}}$ was transmitted. (If the number of ones is equal to the number of zeroes, then we have a tie and it does not matter what we decide.) To prove this formally we note that the ML decides in favor of $\sqrt{\mathcal{E}}$ if the Likelihood Ratio is larger than 1:

$$\frac{(1-\alpha-\beta)^{k(\boldsymbol{d})}\alpha^{\ell(\boldsymbol{d})}}{(1-\alpha-\beta)^{\ell(\boldsymbol{d})}\alpha^{k(\boldsymbol{d})}} \ge 1$$

Equivalently, the ML decides in favor of $\sqrt{\mathcal{E}}$ if

$$\left(\frac{1-\alpha-\beta}{\alpha}\right)^{k(\boldsymbol{d})} \ge \left(\frac{1-\alpha-\beta}{\alpha}\right)^{\ell(\boldsymbol{d})}.$$

The final step is to notice that no matter how large we select θ, we must have that $1-\alpha-\beta \geq \alpha$, hence $\frac{1-\alpha-\beta}{\alpha} \geq 1$. This can be seen both from a sketch of the Gaussian bell and from the following simple analysis:

$$\begin{aligned} 1-\beta-\alpha &= 1-Q\left((1-\theta)\sqrt{\frac{2\mathcal{E}}{N_0}}\right)+Q\left((1+\theta)\sqrt{\frac{2\mathcal{E}}{N_0}}\right)-Q\left((1+\theta)\sqrt{\frac{2\mathcal{E}}{N_0}}\right) \\ &= Q\left((\theta-1)\sqrt{\frac{2\mathcal{E}}{N_0}}\right) \overset{(a)}{\geq} Q\left((\theta+1)\sqrt{\frac{2\mathcal{E}}{N_0}}\right) = \alpha \end{aligned}$$

where (a) follows from the fact that $(\theta-1)\sqrt{\frac{2\mathcal{E}}{N_0}} \leq (\theta+1)\sqrt{\frac{2\mathcal{E}}{N_0}}$.
But this means that the ML decides in favor of $\sqrt{\mathcal{E}}$ if

$$k(\boldsymbol{d}) \;\geq\; \ell(\boldsymbol{d}),$$

that is, if there are at least as many ones in the sequence $\boldsymbol{d}$ as there are zeroes.