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Handout 15 Advanced Digital Communications
Solutions to Homework 6 Nov. 7, 2016

Solution 1.

(a) There are various ways to see what conditions should $\boldsymbol{\mu}$ and ρ satisfy, the most straightforward way is to write out the PDF of $f_Z(z)$ explicitly. Substitute X_1 with $\Re(z)$ and X_2 with $\Im(z)$ in the PDF and carry out the matrix inversion, we have

$$f_Z(\Re(z),\Im(z)) = \frac{1}{2\pi\sigma_{\Re(z)}\sigma_{\Im(z)}\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(\Re(z)-\mathbb{E}[\Re(z)])^2}{\sigma^2_{\Re(z)}}\right.\right.$$
$$\left.\left.+\frac{(\Im(z)-\mathbb{E}[\Im(z)])^2}{\sigma^2_{\Im(z)}}-\frac{2\rho(\Re(z)-\mathbb{E}[\Re(z)](\Im(z)-\mathbb{E}[\Im(z)]}{\sigma_{\Re(z)}\sigma_{\Im(z)}}\right]\right)$$

With the assumption that $\sigma^2_{\Re(z)}=\sigma^2_{\Im(z)}=N_0/2$, the above expression simplifies to

$$f_Z(\Re(z),\Im(z)) = \frac{1}{\pi N_0\sqrt{1-\rho^2}}$$
$$\cdot\exp\left(-\frac{(\Re(z)-\mathbb{E}[\Re(z)])^2+(\Im(z)-\mathbb{E}[\Im(z)])^2-2\rho(\Re(z)-\mathbb{E}[\Re(z)](\Im(z)-\mathbb{E}[\Im(z)]}{N_0(1-\rho^2)}\right)$$

Now by inspection it is easy to see that if $\rho=0$ and $\mathbb{E}[\Re(z)]=\mathbb{E}[\Im(z)]=0$, or equivalently $\boldsymbol{\mu}=[0,0]$, we will have

$$\begin{aligned}f_Z(\Re(z),\Im(z)) &= \frac{1}{\pi N_0}\exp\left(-\frac{\Re(z)^2+\Im(z)^2}{N_0}\right)\\ &= \frac{1}{\pi N_0}e^{-\frac{1}{N_0}|z|^2}\end{aligned}$$

Also notice that if either of the three conditions, i.e., $\sigma_{\Re(z)}=\sigma_{\Im(z)}$, $\mathbb{E}[\Re(z)]=\mathbb{E}[\Im(z)]=0$, $\mathbb{E}[\Re(z)\Im(z)]=0$ is missing, the PDF of Z *cannot* take the above simple form.

(b) Rewrite V as

$$V=(\cos\theta+j\sin\theta)(X_1+jX_2)=X_1\cos\theta-X_2\sin\theta+j(X_1\cos\theta+X_2\sin\theta)$$

Due to the circularly symmetric assumption on Z, random variables X_1,X_2 are two i.i.d. Gaussian random variable with mean zero and variance, say $N_0/2$. Also notice that the real and imaginary part of V are Gaussian random variables, since they are linear combinations of Gaussian random variables X_1,X_2. Now we calculate the mean and variance of the real and imaginary part of V.

$$\begin{aligned}\mathbb{E}[\Re(V)] &= \mathbb{E}[X_1\cos\theta-X_2\sin\theta]=\mathbb{E}[X_1]\cos\theta-\mathbb{E}[X_2]\sin\theta=0\\ \mathbb{E}[\Im(V)] &= \mathbb{E}[X_1\cos\theta+X_2\sin\theta]=\mathbb{E}[X_1]\cos\theta+\mathbb{E}[X_2]\sin\theta=0\\ \mathbb{E}[(\Re(V)-\mathbb{E}[\Re(V)])^2] &= \mathbb{E}[X_1^2]\cos^2\theta+\mathbb{E}[X_2^2]\sin^2\theta-2\cos\theta\sin\theta\,\mathbb{E}[X_1X_2]=N_0/2\\ \mathbb{E}[(\Im(V)-\mathbb{E}[\Im(V)])^2] &= \mathbb{E}[X_1^2]\cos^2\theta+\mathbb{E}[X_2^2]\sin^2\theta+2\cos\theta\sin\theta\,\mathbb{E}[X_1X_2]=N_0/2\end{aligned}$$

where we have used the fact $\mathbb{E}[X_1X_2] = \mathbb{E}[X_1]\,\mathbb{E}[X_2] = 0$ and $\sin^2\theta + \cos^2\theta = 1$. Furthermore, the real and imaginary part are uncorrelated as

$$\begin{aligned}
\mathbb{E}[\Re(V)\Im(V)] &= \mathbb{E}[X_1X_2]\cos^2\theta - \mathbb{E}[X_1X_2]\sin^2\theta + \mathbb{E}[X_1^2]\sin\theta\cos\theta - \mathbb{E}[X_2^2]\sin\theta\cos\theta \\
&= \mathbb{E}[X_1^2]\sin\theta\cos\theta - \mathbb{E}[X_2^2]\sin\theta\cos\theta \\
&= N_0/2(\sin\theta\cos\theta - \sin\theta\cos\theta) \\
&= 0
\end{aligned}$$

Hence we see that the real and imaginary part of V are Gaussian random variables which also satisfy the three conditions, i.e., $\sigma_{\Re(V)} = \sigma_{\Im(V)}$, $\mathbb{E}[\Re(V)] = \mathbb{E}[\Im(V)] = 0$, $\mathbb{E}[\Re(V)\Im(V)] = 0$, hence V is circularly symmetric. Furthermore, the real and imaginary part also have variance $N_0/2$, the same as Z. We conclude that the PDF of V is the same as in part (a).

(c) Use the same argument as in part (b), we can show that the mean and variance of real and imaginary part of W satisfy

$$\begin{aligned}
\mathbb{E}[\Re(W)] &= 0 \\
\mathbb{E}[\Im(W)] &= 0 \\
\mathbb{E}[(\Re(W) - \mathbb{E}[\Re(W)])^2] &= |a|^2N_0/2 \\
\mathbb{E}[(\Im(W) - \mathbb{E}[\Im(W)])^2] &= |a|^2N_0/2
\end{aligned}$$

and

$$\mathbb{E}[\Re(W)\Im(W)] = 0$$

Hence W is also a circularly symmetric Gaussian variable with PDF

$$f_W(w) = \frac{1}{\pi|a|^2N_0}e^{-\frac{1}{N_0|a|^2}|w|^2}$$

Solution 2.

(a) Since everything is symmetric we estimate A if $Y[k] > 0$ and $-A$ if $Y[k] \leq 0$. The error probability of such estimator is:

$$\begin{aligned}
P_e = \Pr\{X[k] = X[k-1]\}\cdot\Pr\{Z[k] > 2A\} + \Pr\{X[k] \neq X[k-1]\}\cdot\Pr\{Z[k] > 0\} \\
= \frac{1}{2}Q\left(\frac{2A}{\sigma}\right) + \frac{1}{4}
\end{aligned}$$

(b) The best decision rule is to estimate A if $Y[k]-\hat{X}[k-1] > 0$ and A if $Y[k]-\hat{X}[k-1] \leq 0$. The error probability of the estimator is now computed as follows. First note that

$$\Pr\{\hat{X}[k] \neq X[k] | \hat{X}[k-1] = X[k-1]\} = Q\Big(\frac{A}{\sigma}\Big).$$

Furthermore,

$$\begin{aligned}
&\Pr\{\hat{X}[k] \neq X[k] | \hat{X}[k-1] \neq X[k-1]\} \\
&\quad = \Pr\{\hat{X}[k] \neq X[k] | \hat{X}[k-1] \neq X[k-1], X[k] = X[k-1]\}\Pr\{X[k] = X[k-1]\} \\
&\qquad + \Pr\{\hat{X}[k] \neq X[k] | \hat{X}[k-1] \neq X[k-1], X[k] \neq X[k-1]\}\Pr\{X[k] \neq X[k-1]\} \\
&\quad = \frac{1}{2}Q\Big(\frac{3A}{\sigma}\Big) + \frac{1}{2}Q\Big(\frac{-A}{\sigma}\Big).
\end{aligned}$$

Consequently

$$\begin{aligned}
P_e &= \Pr\{\hat{X}[k-1] = X[k-1]\} \cdot \Pr\{\hat{X}[k] \neq X[k] | \hat{X}[k-1] = X[k-1]\} \\
&\quad + \Pr\{\hat{X}[k-1] \neq X[k-1]\} \cdot \Pr\{\hat{X}[k] \neq X[k] | \hat{X}[k-1] \neq X[k-1]\} \\
&= (1-P_e) Q\Big(\frac{A}{\sigma}\Big) + P_e \Big[\frac{1}{2} Q\Big(\frac{3A}{\sigma}\Big) + \frac{1}{2} Q\Big(\frac{-A}{\sigma}\Big)\Big]
\end{aligned}$$

By solving the equation we obtain:

$$P_e = \frac{2Q\left(\frac{A}{\sigma}\right)}{1 + 3Q\left(\frac{A}{\sigma}\right) - Q\left(\frac{3A}{\sigma}\right)}$$

We can observe that as A increases the probability of error goes to 0 while without feedback the error probability is lower-bounded by $\frac{1}{4}$.

SOLUTION 3.

(a) The length of one OFDM symbol is

$$N + \text{length of cyclic prefix} = N + L - 1$$

which equals 9 for $L = N = 5$.

The equivalent system has the matrix representation

$$\begin{pmatrix} Y[0] \\ Y[1] \\ Y[2] \\ Y[3] \\ Y[4] \end{pmatrix} = \begin{pmatrix} h_0 & h_4 & h_3 & h_2 & h_1 \\ h_1 & h_0 & h_4 & h_3 & h_2 \\ h_2 & h_1 & h_0 & h_4 & h_3 \\ h_3 & h_2 & h_1 & h_0 & h_4 \\ h_4 & h_3 & h_2 & h_1 & h_0 \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} + \begin{pmatrix} W[0] \\ W[1] \\ W[2] \\ W[3] \\ W[4] \end{pmatrix}$$

For the case when $N = 7$, the length of one OFDM symbol is 11, and the matrix representation is

$$\begin{pmatrix} Y[0] \\ Y[1] \\ Y[2] \\ Y[3] \\ Y[4] \\ Y[5] \\ Y[6] \end{pmatrix} = \begin{pmatrix} h_0 & 0 & 0 & h_4 & h_3 & h_2 & h_1 \\ h_1 & h_0 & 0 & 0 & h_4 & h_3 & h_2 \\ h_2 & h_1 & h_0 & 0 & 0 & h_4 & h_3 \\ h_3 & h_2 & h_1 & h_0 & 0 & 0 & h_4 \\ h_4 & h_3 & h_2 & h_1 & h_0 & 0 & 0 \\ 0 & h_4 & h_3 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & h_4 & h_3 & h_2 & h_1 & h_0 \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{pmatrix} + \begin{pmatrix} W[0] \\ W[1] \\ W[2] \\ W[3] \\ W[4] \\ W[5] \\ W[6] \end{pmatrix}$$

(b) We need to check that $H\boldsymbol{u} = \lambda \boldsymbol{u}$. Indeed, for all $1 \le n \le N$, the n^{th} entry of $H\boldsymbol{u}$ is

$$\begin{aligned}
(H\boldsymbol{u})_n &= \sum_{k=1}^{N} (H)_{nk} (\boldsymbol{u})_k = \sum_{k=1}^{n} h_{n-k} \alpha^{1-k} + \sum_{k=n+1}^{N} h_{N+n-k} \alpha^{1-k} \\
&= \sum_{l=0}^{n-1} h_l \alpha^{l+(1-n)} + \sum_{l=n}^{N-1} h_l \alpha^{l+(1-n)-N} \\
&\overset{(*)}{=} \sum_{l=0}^{N-1} h_l \alpha^{l+(1-n)} = \left(\sum_{l=0}^{N-1} h_l \alpha^l \right) \alpha^{1-n} = \lambda (\boldsymbol{u})_n,
\end{aligned}$$

where $(*)$ follows since $\alpha^N = 1$.

Solution 4.

(a) The fastest way to proceed is to use the Parseval's identity:

$$P_{average} = \frac{1}{N}\mathbb{E}\left[\|\boldsymbol{x}\|^2\right] = \frac{1}{N}\mathbb{E}\left[\|\boldsymbol{X}\|^2\right].$$

It is perhaps good to write out the norm explicitly:

$$\begin{aligned} P_{average} &= \frac{1}{N}\mathbb{E}\left[\|\boldsymbol{X}\|^2\right] = \frac{1}{N}\mathbb{E}\left[\sum_{i=0}^{N-1}|X_i|^2\right] = \frac{1}{N}\sum_{i=0}^{N-1}\mathbb{E}\left[|X_i|^2\right] \\ &= \frac{1}{N}\sum_{i=0}^{N-1}\mathcal{E} = \mathcal{E}. \end{aligned}$$

(b) This part was a little more difficult than (a). By observing that the first row of the inverse Fourier matrix is simply the all-ones vector (scaled by $1/\sqrt{N}$), we can write the element

$$x_0 = \frac{1}{\sqrt{N}}\sum_{i=0}^{N} X_i.$$

If we set all $X_i = \sqrt{\mathcal{E}}$, the corresponding $x_0 = \sqrt{N}\sqrt{\mathcal{E}}$, and hence, $|x_0|^2 = N\mathcal{E}$. So, the maximum is at least $N\mathcal{E}$. Could it be larger? The answer is no. A fast way to get this answer is to observe that $x_i = \boldsymbol{f}_i^H\boldsymbol{X}$, for $i = 0, 1, \ldots, N-1$, where $\boldsymbol{f}_i^H$ denotes the $(i+1)$th row of the inverse Fourier matrix. By the Cauchy–Schwarz inequality, we find

$$|x_i|^2 = |\boldsymbol{f}_i^H\boldsymbol{X}|^2 \le \|\boldsymbol{f}_i\|^2\|\boldsymbol{X}\|^2,$$

and clearly, $\|\boldsymbol{X}\|^2 = N\mathcal{E}$. Moreover, the rows and columns of the Fourier matrix are all normalized, hence $\|\boldsymbol{f}_i\| = 1$. Hence, $|x_i|^2 \le N\mathcal{E}$.

(c) As we have seen,

$$PAPR = \frac{P_{peak}}{P_{average}} = \frac{N\mathcal{E}}{\mathcal{E}} = N.$$

One of the issues with such a PAPR is the following: In practice the transmitted signal has to pass through amplifiers that behave linearly only in a certain dynamic range of the input signal. Now if the OFDM length N gets larger and larger we see that the peak power increases which means the chance of the signal exceeding the "linearity thresholds" of the components and, hence, driving them into the non-linear phase (and hence introducing distortion) gets higher and higher.

Solution 5.

(a) The general optimal rate allocation problem has the form

$$\begin{aligned} \text{minimize} \quad & -\sum_{i=1}^{K}\alpha_i \log_2(1+\beta_i P_i) \\ \text{subject to} \quad & -P_i \le 0, \ \text{for } i = 1, 2, \ldots, K, \\ & \sum_{i=1}^{K} P_i - P = 0. \end{aligned}$$

with P_i, $i = 1, \ldots, K$ the variables.

(b) The Lagrangian for this problem is

$$L(\{P_i\}, \{\lambda_i\}, \nu) = -\sum_{i=1}^{K} \alpha_i \log_2 (1 + \beta_i P_i) - \sum_{i=1}^{K} \lambda_i P_i + (\sum_{i=1}^{K} P_i - P)\nu$$

(c) The KKT conditions for this problem are:

$$\begin{aligned}
(\log_2 e)\frac{\alpha_i \beta_i}{1 + \beta_i P_i} &= \nu - \lambda_i, \quad i = 1, \ldots, K, \\
-P_i &\leq 0, \quad i = 1, \ldots, K, \\
\sum_{i=1}^{K} P_i - P &= 0, \\
\lambda_i &\geq 0, \quad i = 1, \ldots, K, \\
\lambda_i P_i &= 0, \quad i = 1, \ldots, K.
\end{aligned}$$

(d) The last condition above implies we either have $P_i = 0$ or $\lambda_i = 0$. If $\lambda_i = 0$, the first condition can be written as

$$\frac{\alpha_i \beta_i}{1 + \beta_i P_i} = \nu \ln 2.$$

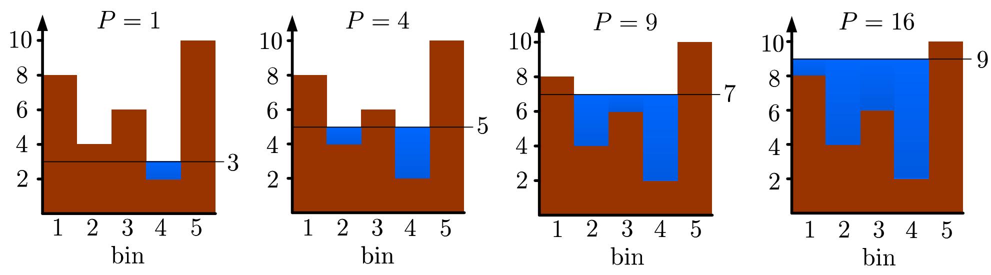
(e) Assuming that $\alpha_i = 1$ and $\beta_i = \frac{|H_i|^2}{N_0}$, the result in (d) becomes

$$\begin{aligned}
\text{Either } P_i \text{ satisfies:} \quad & \frac{|H_i|^2}{N_0 + |H_i|^2 P_i} = \nu \ln 2 \\
\text{or} \quad & P_i = 0.
\end{aligned}$$

or equivalently

$$\begin{aligned}
\text{Either } P_i \text{ satisfies:} \quad & P_i + \frac{N_0}{|H_i|^2} = \frac{\log_2 e}{\nu} \\
\text{or} \quad & P_i = 0.
\end{aligned}$$

(f) Think of $N_i = \frac{N_0}{|H_i|^2}$ as bins and P the amount of water we can fill. The figure below shows the solutions of water-filling for $P = 1, 4, 9, 16$, respectively.



Solution 6.

(a)

$$\boldsymbol{y} = H\boldsymbol{x} + \boldsymbol{z}$$

$$F_N \boldsymbol{y} = F_N S H \boldsymbol{x} + F_N \boldsymbol{z}$$

$$\Rightarrow \quad \boldsymbol{Y} = F_N S F_N^{-1} D F_N \boldsymbol{x} + F_N \boldsymbol{z}$$
$$\Rightarrow \quad \boldsymbol{Y} = \underbrace{F_N S F_N^{-1} D}_{G} \boldsymbol{X} + \boldsymbol{Z}$$
$$\Rightarrow \quad \boldsymbol{Y} = G\boldsymbol{X} + \boldsymbol{Z}$$

(b)

$$Y[l] = G_{l,l}X[l] + \underbrace{\sum_{q\neq l} G_{l,q}X[q] + Z[l]}_{\text{ICI + noise}}, \quad l = 0, \ldots, N-1,$$

Hence,

$$\text{SINR} = \frac{\mathbb{E}[|G_{l,l}X[l]|^2]}{\mathbb{E}[\left|\sum_{q\neq l} G_{l,q}X[q]\right|^2] + \mathbb{E}\,|Z_l|^2} = \frac{\mathcal{E}|G_{l,l}|^2}{\mathcal{E}\sum_{q\neq l}|G_{l,q}|^2 + \sigma_z^2}$$

(c)

$$\begin{aligned}
\mathbb{E}[\boldsymbol{Y}\boldsymbol{Y}^H] &= \mathbb{E}[(G\boldsymbol{X} + \boldsymbol{Z})(\boldsymbol{X}^H G^H + \boldsymbol{Z}^*)] \\
&= \mathcal{E}GG^H + \sigma_z^2 I. && (1) \\
\mathbb{E}[\boldsymbol{X}\boldsymbol{Y}^H] &= \mathbb{E}[\boldsymbol{X}(\boldsymbol{X}^H G^H + \boldsymbol{Z}^H)] \\
&= \mathcal{E}G^H, && (2)
\end{aligned}$$

Orthogonality principle implies,

$$\mathbb{E}[(W^H\boldsymbol{Y} - \boldsymbol{X})\boldsymbol{Y}^H] = 0$$

That is,

$$\mathbb{E}[W^H\boldsymbol{Y}\boldsymbol{Y}^H] = \mathbb{E}[\boldsymbol{X}\boldsymbol{Y}^H] \quad \Rightarrow \quad W^H = \mathbb{E}[\boldsymbol{X}\boldsymbol{Y}^H](\mathbb{E}[\boldsymbol{Y}\boldsymbol{Y}^H])^{-1}.$$

Using equations (1) and (2) we get that,

$$W^H = \mathcal{E}G^H(\mathcal{E}GG^H + I\sigma_z^2)^{-1}.$$