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Handout 2 Advanced Digital Communications
Quiz Sep. 21, 2016

PROBLEM 1. Consider the Discrete Fourier Transform of length N given by the pair

$$X[n] = \mathcal{F}\{x\} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x[k] e^{-j\frac{2\pi}{N}kn}, \qquad x[n] = \mathcal{F}^{-1}\{X\} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}2kn}.$$

Show the following properties:

(a) $\mathcal{F}\{\mathcal{F}^{-1}\{X\}\} = X$.

(b) $\mathcal{F}\{x \circledast y\} = \sqrt{N}\mathcal{F}\{x\}\mathcal{F}[y]$ (where $(x \circledast y)[n] = \sum_{m=0}^{N-1} x[m]y[(n-m) \bmod N]$).

(c) If $y[n] = x[(n+m) \bmod N]$, then $\mathcal{F}\{y\}[n] = X[n]e^{j\frac{2\pi}{N}nm}$.

(d) $\sum_{n=0}^{N-1} x^*[n]y[n] = \sum_{n=0}^{N-1} X^*[n]Y[n]$.

PROBLEM 2.

(a) Your take a medical test for Dysania, a potentially serious illness. The test is 99% accurate, i.e., the result of the test is with probability 0.99 correct and with probability 0.01 false. A priori, the chance of having Dysania is one in one million. The test comes out positive. Should you be worried?

(b) A random variable is:

  (i) A coin that has two faces and can be flipped?
  (ii) A mapping from the space of outcomes to the reals?
  (iii) A variable that is not fixed but random?
  (iv) A probability distribution?

(c) Write down the density of a Gaussian random variable with mean -1 and variance 3. What is the chance that a random variable X drawn from this distribution takes on a value larger than 4? Write down your answer in terms of the standard Q function. For large positive values x, how does the probability that X exceeds x decay as a function of x?

(d) Let X be a non-negative random variable with mean $\bar{X}$. What is the best bound for $\Pr\{X \geq \alpha\}$, for $\alpha > 0$ given only this information?

(e) Consider the Gaussian vector of length 3, (X_1, X_2, X_3) with mean $(0, 1, -3)$ and covariance matrix

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & 2 & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & 3 \end{pmatrix}$$

What is the expected value of the sum of the three components?

(f) Consider n events A_i, $i = 1, \cdots, n$. Show that

$$\sum_{i=1}^{n} \mathbb{P}\{A_i\} - \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbb{P}\{A_i \cap A_j\} \leq \mathbb{P}\Big\{\bigcup_{i=1}^{n} A_i\Big\} \leq \sum_{i=1}^{n} \mathbb{P}\{A_i\}.$$

Problem 3. In a binary communication system, the transmitter sends $x_1 \in \mathbb{R}$ or $x_2 \in \mathbb{R}$, where $x_2 > x_1$, via a discrete-time AWGN channel with noise variance σ^2, namely the channel that maps its input $X \in \mathbb{R}$ to its output Y as

$$X \mapsto Y = X + Z, \qquad Z \sim \mathcal{N}(0, \sigma^2).$$

(a) Assume x_1 and x_2 are equally probably to be chosen. What is the optimal decision rule for the decoder?
Hint. A decision rule is a mapping from the observation space (in this case $\mathbb{R}$ as Y takes values in reals), to the space of hypotheses (in this case $\{x_1, x_2\}$), which we denote by $\hat{X}(y)$. The optimal decision rule, is the one that minimizes the probability of error, $\Pr\{\hat{X}(Y) \neq X\}$.

(b) Now assume x_1 is chosen with probability $p \in [0, 1]$. What is the optimal decision rule?

(c) Compute the probability of error for the receiver of part (a).

Suppose now that repetition coding is used to improve the reliability of communication. More precisely, either

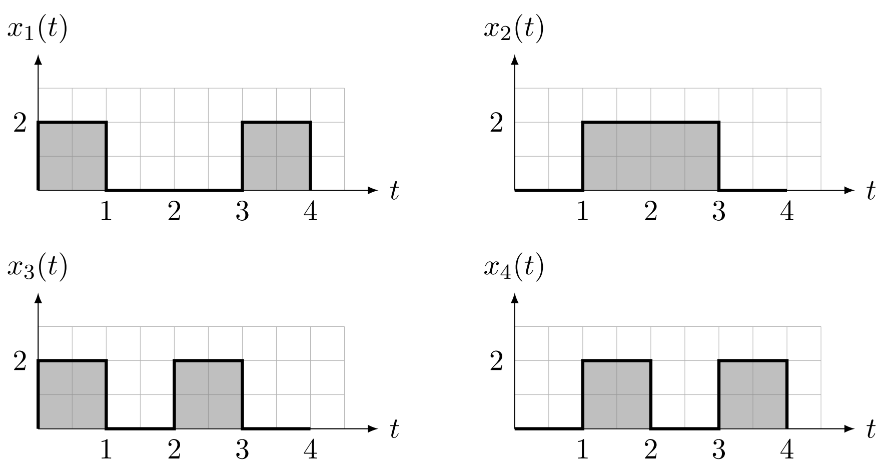
$$\mathbf{x}_1 = (\underbrace{x_1, x_1, \ldots, x_1}_{n \text{ times}}) \in \mathbb{R}^n \qquad \text{or} \qquad \mathbf{x}_2 = (\underbrace{x_2, x_2, \ldots, x_2}_{n \text{ times}}) \in \mathbb{R}^n$$

is transmitted via n independent uses of the channel. Hence, the observable at the receiver is an n-dimensional random vector $\mathbf{Y} = (Y_1, \ldots, Y_n) \in \mathbb{R}^n$.

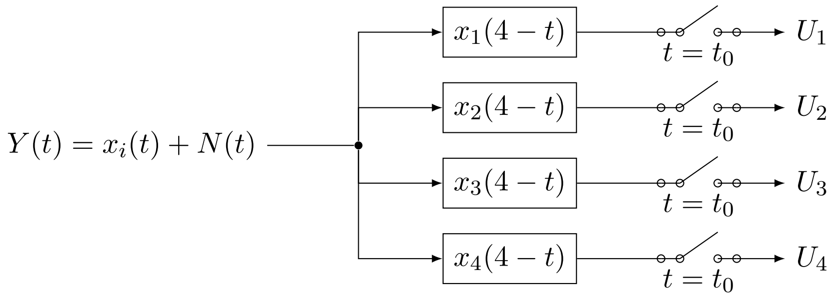
(d) Derive a one-dimensional sufficient statistic for the decision problem at the decoder.

(e) Find the optimal decision rule for equally probable hypotheses and compute its error probability. How does the probability of error scale with n?

Problem 4. Consider the signal set shown below. Each signal is equally likely to be chosen for transmission over a continuous-time additive white Gaussian noise channel with power spectral density $\frac{N_0}{2}$.



The receiver passes the received signal $Y(t) = x_i(t) + N(t)$ through four filters with impulse response $h_i(t) = x_i(4 - t)$, $i = 1, 2, 3, 4$, and then samples their output at time $t = t_0$ to form decision statistics as shown below:



(a) Find the correct sampling time t_0 so that $(U_1, \ldots, U_4)$ forms a sufficient statistic for optimal decision at the decoder.

(b) Describe the optimal decision rule in terms of $(U_1, \ldots, U_4)$.

(c) Represent the signal set using the four basis signals given by $\varphi_1(t) = \varphi(t)$, $\varphi_2(t) = \varphi(t-1)$, $\varphi_3(t) = \varphi(t-2)$, $\varphi_4(t) = \varphi(t-3)$, where

$$\varphi(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(d) Use the union bound to find an upper bound to the error probability of the receiver you found in (b).

(e) Re-implement the receiver using a single filter with impulse response $\varphi(t)$.
Hint. You will need to sample the filter output at multiple time instants to obtain a sufficient statistics for the decision problem.

(f) What is the exact error probability of the receiver?
Hint. Transform the signals by a translation to obtain a minimum energy signal set.