ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 18 Advanced Digital Communications
Midterm Exam Nov. 9, 2016

- 3 problems, 30 points, 105 minutes
- This is a closed book exam.
- Only one double-sided handwritten A4 page of summary allowed.

Good Luck!

PLEASE WRITE YOUR NAME ON EACH SHEET OF YOUR ANSWERS

PLEASE WRITE THE SOLUTION OF EACH PROBLEM ON A SEPARATE SHEET

WE WILL RETURN THE EXAM BACK TO YOU ON MONDAY, NOVEMBER 14 IN CLASS. PLEASE VERIFY THE CORRECTIONS RIGHT AWAY. IN CASE YOU HAVE QUESTIONS, ASK US IN CLASS. ONCE YOU TAKE THE EXAM OUT OF THE CLASS ROOM, WE WILL ASSUME THAT YOU AGREE WITH THE CORRECTIONS.

PROBLEM 1. (10 points)

(a) (8 pts) Consider the problem of detecting $X \in \{-1, +1\}$ given the observable

$$Y = X + Z,$$

where $Z \sim \mathcal{N}(0, \sigma^2)$ is independent of X.

(i) (1 pt) Assume $\Pr\{X = +1\} = \Pr\{X = -1\}$. What is the optimal decision rule?

(ii) (1 pt) Assume now that $\Pr\{X = +1\} = p$. What is the optimal decision rule?

(iii) (1 pt) Compute the error probability of the decision rule of part (ii).

(iv) (2 pts) For what values of p the error probability of the detector of (ii) is *maximum*? (No calculations required for answering this question — but you need to explain and justify your answer.)

In the above we assumed the decoder knows the priors on X. In practice this may not always be possible. In the following, we derive a decoder that minimizes the worst-case probability of error in such situations.

(v) (1 pt) Suppose the decoder implements the optimal decision rule of (ii) assuming $\Pr\{X = +1\} = p$ but in reality $\Pr\{X = +1\} = q$. What is the probability of error of the decoder?

(vi) (1 pt) Denote by $P_e(p, q)$ the probability of error you found in (v). For what values of q, $P_e(p, q)$ is *maximized*?

(vii) (1 pt) Let $P_e(p, *)$ denote the maximum value of $P_e(p, q)$ over the choices of q. (You have already computed this quantity in (vi).) For which value of p, $P_e(p, *)$ is *minimized*? Call this value $p^\star$ and argue that the decoding rule implemented assuming $\Pr\{X = +1\} = p^\star$ is optimal in the sense that it minimizes the worst-case probability of error.

(b) (2 pts) Assume $\boldsymbol{X} \in \{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$ where $\boldsymbol{x}_1 = (1, 0), \boldsymbol{x}_2 = (0, 1)$, and $\boldsymbol{x}_3 = (0, -1)$. Consider the problem of detecting $\boldsymbol{X}$ given the observable

$$\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{Z},$$

where $\boldsymbol{Z}$ is a Gaussian vector with i.i.d. components of zero mean and variance σ^2.

(i) (1 pt) Sketch the decision regions for the ML decoder.

(ii) (1 pt) Using the union bound, find an upper bound on the probability of error of the ML detector.

PROBLEM 2. (10 points) Assume we are given the following discrete time channel model

$$U[n] = \frac{5}{4}I[n] - \frac{1}{2}I[n+1] - \frac{1}{2}I[n-1] + V[n],$$

where the $I[n]$ (information symbols) are i.i.d. with zero mean and $\mathbb{E}[|I[n]|^2] = 1$ taking values in some discrete alphabet $\mathcal{A}$, and $V[n]$ is a zero-mean circularly symmetric Gaussian random process with correlation

$$R_V[k] = -\frac{1}{2}\delta[k-1] + \frac{5}{4}\delta[k] - \frac{1}{2}\delta[k+1].$$

(Recall that $\delta[n]$ is the *unit impulse* defined as $\delta[0] = 1$ and $\delta[n] = 0$ for $n \neq 0$.)

(a) (2 pts) Write down the whitening filter and the resulting channel model at the output of the whitening filter. Note that the whitening filter should be chosen such that the effective channel after filtering be causal. Do not forget to specify the signal *and* the noise.

(b) (1 pt) If at the output of the whitening filter we used the Viterbi algorithm to find the most likely transmitted sequence, what would the required complexity be?

For the remainder of the problem assume instead of whitening the noise and using the Viterbi algorithm we want to filter the channel output $U[n]$ through an equalizer to combat the ISI.

(c) (1 pt) What is the frequency response of the zero-forcing filter in this case?

(d) (1 pt) Compute the effective noise variance after the zero-forcing filtering, namely,

$$\sigma^2_{\text{ZF}} = \mathbb{E}[|I[n] - \hat{I}_{\text{ZF}}[n]|^2],$$

where $\hat{I}_{\text{ZF}}[n]$ is the output of the zero-forcing filter.

(e) (1 pt) What is the frequency response of the LMMSE filter in this case?

(f) (2 pts) Compute the effective noise variance after the LMMSE filtering, namely,

$$\sigma^2_{\text{LMMSE}} = \mathbb{E}[|I[n] - \hat{I}_{\text{LMMSE}}[n]|^2],$$

where $\hat{I}_{\text{LMMSE}}[n]$ is the output of the linear minimum mean-squared error equalizer.

(g) (2 pts) Assume now that the transmitted symbols $I[n]$ are from the alphabet $\{\pm 1\}$. Assume further that at the output of the two equalizers we use a decision device which decides upon the transmitted symbols depending on whether the output is positive or negative. Finally, assume that the "noise" at the input of these decision devices is Gaussian. Which of the two cases results in a smaller probability of error? Is there any other linear equalizer that yields a smaller probability of error? (Almost no calculations required for this question.)

Some useful integrals:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\frac{5}{4} - \cos(2\pi f)} \mathrm{d}f = \frac{4}{3}. \qquad \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\frac{9}{4} - \cos(2\pi f)} \mathrm{d}f = \frac{4}{\sqrt{65}}.$$

PROBLEM 3. (10 points) Consider the following channel model:

$$U[n] = \frac{3}{4}I[n] + \frac{1}{4}I[n-1] + \frac{3}{4}I[n-2] + \frac{1}{4}I[n-3] + Z[n],$$

where $Z[n]$ is circularly symmetric Gaussian noise with variance 1.

We want to transmit information using an OFDM system with $N = 4$.

(a) (1 pt) What is the minimum length of the cyclic prefix (guard interval) that you need to use?

(b) (1 pt) Consider a block that starts at time 0 and goes to time $N - 1 = 3$. Assume that we transmit at these 4 positions the values $x[0], x[1], x[2], x[3]$. What values do we have to assign to $x[-1], \cdots$?

(c) (3 pts) Describe the N parallel channels that you get (channel gain, noise variance). (Note the factor $\sqrt{N}$ in the formula for the convolution! See the reminder!)

(d) (1 pt) Bring each of the N parallel channels into a standard form where the channel gain is 1. What is the equivalent noise variance for each channel?

(e) (2 pts) Suppose we have a total power budget of 11/4 per OFDM block (of length N). What is the optimal allocation?

(f) (2 pts) What is the highest number of bits that can be reliably transmitted with this scheme measured per channel use?

(g) (1 pt) If you were the system designer, what quantity would you change in order to improve the maximum achievable transmission rate.

Recall. Let $x[n]$ be a signal of length N and let $X[n]$ be its DFT. Let $x \circledast y$ denote the cyclic convolution of the two signals x and y. Then we have

$$X[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x[k] e^{-2\pi j \frac{kn}{N}}, \qquad x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] e^{2\pi j \frac{kn}{N}}, \qquad x \circledast y \longleftrightarrow \sqrt{N} XY.$$

Hint. All numbers are extremely simple. If you get complicated numbers go back to square one!