PROBLEM 1. Consider transmission over an ISI channel where the output after matched filtering is

$$U[n] = \sum_{k=-\infty}^{\infty} d[k]I[n-k] + V[n]$$

and $d[k]$ is given by

$$d[k] = \begin{cases} 2^{-\frac{|k|-1}{2}} & \text{if } k \text{ is odd,} \\ \frac{5}{3}2^{-\frac{|k|}{2}} & \text{if } k \text{ is even,} \end{cases}$$

and, as we already know from the lecture, the noise $V[n]$ has autocorrelation function $R_V[k] = \frac{N_0}{2}d[-k]$. Find the filter $d_W[k]$ to whiten the noise. Choose the whitening filter such that the resulting communication channel after the whitening filter is causal.

PROBLEM 2. Consider the discrete-time equivalent of a band-limited AWGN channel with noise power $N_0/2 = 1$. The equivalent channel is a 3-tap channel defined as

$$U[n] = \beta I[n+1] + \alpha I[n] + \beta I[n-1] + V[n].$$

The Gaussian noise $V[n]$ has autocorrelation function

$$R_V[k] = d[-k] = \alpha\delta[k] + \beta\delta[k-1] + \beta\delta[k+1],$$

and α and β are positive real coefficients satisfying $\alpha^2 > 4\beta^2$. We also assume that the information symbols $I[n]$ are i.i.d. random variables with mean 0 and variance $\mathcal{E}$.

(a) Use the zero-forcing equalizer to remove all of the inter-symbol interference. Find the frequency response of the filter and calculate the variance of the effective noise $\tilde{V}[n] = I[n] - \hat{I}_{ZF}[n]$ (see Equation (4.13) of your lecture notes).

(b) Now apply the LMMSE approach. Find the frequency response of LMMSE filter $A_{\text{LMMSE}}(f)$. Calculate the resulting noise variance σ^2_{LMMSE} (as in Equation (4.15) of your lecture notes) using the following steps:

   (i) Show that, in general, $\sigma^2_{\text{LMMSE}} = \mathbb{E}[I[n]^2] - \mathbb{E}[\hat{I}_{\text{LMMSE}}[n]I[n]]$.

   (ii) Use (i) to find σ^2_{LMMSE} for the particular example considered in the problem.

(c) Another tempting receiver the matched filter: It passes the channel output signal through a filter whose impulse response is the time-reversed version of the original channel impulse response. Show that the corresponding output signal can be expressed as

$$I_{\text{MF}}[n] = g[0]I[n] + \sum_{k\neq 0} g[k]I[n-k] + \sum_m f[m]V[n-m],$$

give the values of the coefficients $g[k]$, and $f[m]$, and calculate the effective noise variance, that is, the variance of the $\frac{1}{g[0]}I_{\text{MF}}[n] - I[n]$.

(d) For $\alpha = 1$ and $\beta = 0.4$, plot the effective noise variances you found in (a)–(c) for the range of $\mathcal{E} \in [-20 \text{ dB}, +20 \text{ dB}]$ and compare them.

Hint. You may need the following integrals: For $a \geq |b|$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{dx}{a + b\cos x} = \frac{1}{\sqrt{a^2 - b^2}}, \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{dx}{(a + b\cos x)^2} = \frac{a}{(a^2 - b^2)\sqrt{a^2 - b^2}}.$$

PROBLEM 3. Consider the following *two user* discrete-time channel

$$Y[n] = I[n] - 2I[n-1] + J[n] + J[n-1] + W[n]$$

where $I[n]$ is the information symbol from user 1 and $J[n]$ is the information symbol from user 2. We assume that the sequences $I[n]$ and $J[n]$ are both i.i.d. zero-mean random variables with unit variance independent of each other, and that $W[n]$ is white Gaussian noise with variance $N_0/2 = 1$. Given the channel output $Y[n]$ we would like to design linear MMSE estimators to detect both $I[n]$ and $J[n]$ in the form

$$\hat{I}[n] = a_{-1}Y[n+1] + a_0 Y[n] + a_1 Y[n-1] \quad \text{and} \quad \hat{J}[n] = b_{-1}Y[n+1] + b_0 Y[n] + b_1 Y[n-1].$$

Compute the optimal coefficients a_i, b_i for $i = -1, 0, 1$ which minimize the mean squared error $\mathbb{E}[(\hat{I}[n] - I[n])^2]$ and $\mathbb{E}[(\hat{J}[n] - J[n])^2]$.

PROBLEM 4. Consider the noisy ISI channel given by

$$Y[n] = X[n] + X[n-1] + Z[n]$$

where $X[n]$ and $Y[n]$ are the channel input and output, respectively at time index i, $Z[n]$ is a sequence of i.i.d. Gaussian random variables, with zero mean and unit variance and $X[n] \in \{-1, 1\}$ uniformly. Calculate the symbol-wise MAP estimate of $X[1], \ldots, X[4]$ using the BCJR algorithm, if the received sequence is $1, -2, -1, 3, 2$. Assume that the channel is in state $+1$ at the beginning and at the end of the sequence (i.e., $X[0] = X[5] = +1$). Compare this to the decoding estimate from MLSE (Viterbi) decoder.

PROBLEM 5. Consider the following real channel,

$$\boldsymbol{Y} = \boldsymbol{h}X + \boldsymbol{Z},$$

where $X \in \mathbb{R}$ is a random variable, with $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^2] = \mathcal{E}$, $\boldsymbol{h}$ is a fixed real (column) vector, and $\boldsymbol{Z}$ is a zero-mean random vector with covariance matrix I, chosen independently of X.

(a) An estimator $\hat{X}(\boldsymbol{y})$ is said to be *unbiased* if $\mathbb{E}[\hat{X}(\boldsymbol{Y})|X = x] = x$.

  (i) What is the constraint for a linear estimator, i.e $\hat{X} = \boldsymbol{a}^T\boldsymbol{Y}$ to be unbiased?

  (ii) Find the unbiased linear estimator that minimizes the mean squared error $\sigma^2_{\text{unbiased}} = \mathbb{E}[(X - \hat{X})^2]$ and the value of $\sigma^2_{\text{unbiased}}$ for this estimator.
  Hint. By Cauchy–Schwartz inequality, $(\boldsymbol{a}^T\boldsymbol{a})(\boldsymbol{h}^T\boldsymbol{h}) \geq |\boldsymbol{a}^T\boldsymbol{h}|^2$.

(b) In this part, we don't restrict ourselves to unbiased estimators. Suppose $\hat{X} = \boldsymbol{a}^t\boldsymbol{Y}$ is a linear estimator. Find the linear estimator that minimizes the mean squared error $\sigma^2 = \mathbb{E}[(X - \hat{X})^2]$ and the value of σ^2 for this estimator.
Hint. First assume $\boldsymbol{a}^T\boldsymbol{h} = c$, and minimize σ^2 with respect to the vector $\boldsymbol{a}$ with the constraint $\boldsymbol{a}^T\boldsymbol{h} = c$, and then minimize the result with respect to c.

(c) Compare the two "signal to noise ratio"s $\mathcal{E}/\sigma^2_{\text{unbiased}}$ and $\mathcal{E}/\sigma^2$.

(d) Now assume X is equally likely to be $+1$ or -1. Suppose a decision is made by quantizing the estimate $\hat{X}$ from either part (a) or (b) to ± 1. Which estimator would you choose to minimize the probability of error?