ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 10 Advanced Digital Communications
Homework 4 Oct. 17, 2016

PROBLEM 1. Consider the i.i.d. random process $X[n]$, with mean zero and unit variance.

(a) Prove explicitly that this random process is stationary and give its autocorrelation function and power spectral density.

(b) Now, suppose that we pass this random process through low-pass filter to obtain the new random process

$$Y[n] = \sum_{k=0}^{\infty}(1/2)^k X[n-k].$$

Prove explicitly that this random process is wide-sense stationary, and give the autocorrelation function and the power spectral density in the frequency domain. Give a sketch of the power spectral density. Does the sketch make sense? Explain in a few sentences.

(c) Repeat (c) for the model

$$Y[n] = X[n] + \frac{1}{3}X[n-1] + Z[n],$$

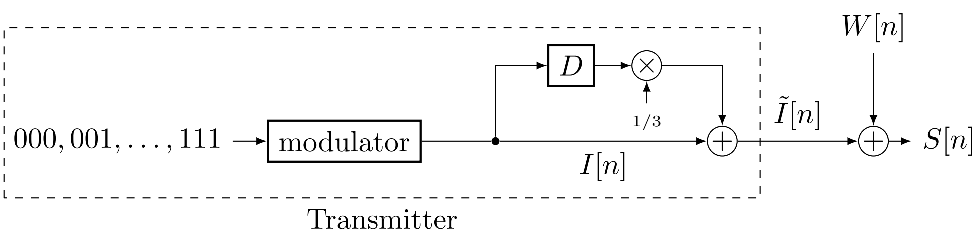
where $Z[n]$ is an i.i.d. Gaussian random process with mean zero and variance σ^2 independent of $X[n]$.

PROBLEM 2. Consider the following channel with inter-symbol interference

$$S[n] = I[n] + \frac{1}{3}I[n-1] + W[n],$$

where $W[n]$ is white Gaussian noise with variance $N_0/2$. Suppose that now we want to send three bits $b_0, b_1, b_2 \in \{0, 1\}$. The information symbols are chosen as $I[n] = (2b_n - 1)\sqrt{\mathcal{E}}$, $n = 0, 1, 2$. Also, we assume that the transmitter sends $I[n] = -\sqrt{\mathcal{E}}$ at time instants $n = -1$ and $n = 3$, which is known at the receiver.

(a) For all possible combinations of b_0, b_1, and b_2, determine the equivalent transmitted signal $\tilde{I}[n], n = 0, 1, 2, 3$, as shown in figure below.



(b) Your answer to (a) is nothing but a signal constellation of 8 signal points, to be transmitted across a standard AWGN channel. Determine the minimum distance of the constellation and give a bound on the average error probability.

Alternatively we can first apply a *zero-forcing filter* on the channel output to eliminate the ISI and perform the detection based on the filtered channel output. More precisely, we know that the output signal $S[n]$ has the generic form of

$$S[n] = \sum_{k=-\infty}^{\infty} f[k]I[n-k] + W[n] = (f * I)[n] + W[n].$$

The zero-forcing filter is nothing but the inverse of the *channel response*, $f[n]$. If we denote its impulse response by $d_{ZF}[n]$, its output to the input $S[n]$ will be

$$\hat{I}_{ZF}[n] = (d_{ZF} * S)[n] = I[n] + \underbrace{(d_{ZF} * W)[n]}_{\tilde{V}[n]}.$$

(See section 4.5.1 of your lecture notes for further details.)

(c) Determine the zero-forcing filter $D_{ZF}(z)$ and the power spectral density of the resulting noise. Then, show that this leads to a noise power of $\mathbb{E}[|\tilde{V}|^2] = \frac{9}{16}N_0$.

(d) Following (c), calculate the error probability of estimating just one symbol incorrectly in terms of the Q-function. It is hard to calculate the total error probability of all three bits. Instead, we will use this single bit error probability as a lower bound. This is justified, because surely the probability of getting multiple bits wrong in a string can never be less than the probability of getting just one bit wrong. Plot and compare this *lower* bound on zero-forcing to the *upper* bound of the optimal ML-detector we found in question (b).

PROBLEM 3. Consider the following $2N$-dimensional real-valued Gaussian vector problem:

$$\boldsymbol{Y} = \boldsymbol{x} + \boldsymbol{Z}$$

where $\boldsymbol{x} \in \mathbb{R}^{2N}$ is selected from $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_M\}$ and $\boldsymbol{Z}$ is a zero-mean Gaussian noise with covariance matrix

$$\Sigma_{\boldsymbol{Z}} = \begin{bmatrix} \sigma_0^2 I_N & 0 \\ 0 & \sigma_1^2 I_N \end{bmatrix}.$$

That is, all components of $\boldsymbol{Z}$ are assumed to be independent, but we assume that the first N components have variance σ_0^2 while the last N have variance σ_1^2. This is due to a suddenly changing channel condition, as is frequent in wireless communication.

(a) Derive the ML detector. Simplify it as much as possible. Finally, work out the case where $\sigma_0^2 = 2$ and $\sigma_1^2 = 1$, in preparation for (b).

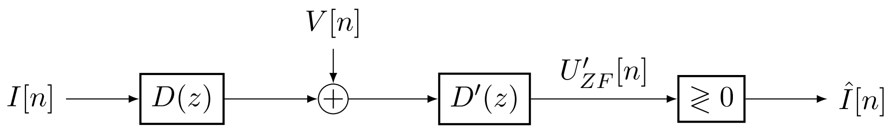
(b) Consider the ISI channel

$$Y[n] = 3I[n] + I[n-1] + Z[n],$$

where all the noises $Z[n]$ are zero-mean Gaussians and independent of each other, but they do *not* all have the same variance. In particular, let us assume that $\mathbb{E}[Z^2[0]] = \mathbb{E}[Z^2[1]] = 2$, but $\mathbb{E}[Z^2[n]] = 1$, for all other n (i.e., $n \geq 2$). Moreover, suppose that $I[n] \in \{-1, +1\}$ are i.i.d. with uniform priors. Given that $I[-1] = -1$ and that the first eight samples of the received sequence are $(0, -1, , 6, 1, -2, -5, 0, 3)$, find the ML estimate of $I[0], I[1], I[2]$ using the Viterbi algorithm.

(c) Give an upper bound on the probability that your ML estimate from (b) is wrong.

PROBLEM 4. The key of zero-forcing equalization is that one counters the ISI by taking the inverse of the filter $d[k]$. In practice, one would have to measure the channel response and use well-fitted models, but these may very well not be exact at all. In this question, though, we look at the effects of processing your channel with the *wrong* filter.



We consider the standard discrete-time equivalent channel that is suffering from ISI:

$$U[n] = \sum_{k=-\infty}^{\infty} I[n-k]d[k] + V[n], \quad D(z) = \frac{(1-\beta z)(1-\beta z^{-1})}{(1-\alpha z)(1-\alpha z^{-1})}$$

and we combat the ISI by the following imperfect zero-forcing filter:

$$D'_{ZF}(z) = (1-\alpha z)(1-\alpha z^{-1}).$$

Assume that we use a binary source $I[n] \in \{\pm\sqrt{\mathcal{E}}\}$ and that the *correlated* noise source V has an autocorrelation $R_V[k] = \frac{N_0}{2}d[-k]$.

(a) Show that $U'_{ZF}[n]$ is of the form

$$U'_{ZF}[n] = f[0]I[n] + \sum_{k=-\infty, k\neq 0}^{k=+\infty} f[k]I[n-k] + V'[n], \qquad \text{with } V'[n] = \sum_{k=-\infty}^{+\infty} g[k]V[n-k]$$

and find expressions for $f[k]$ and $g[k]$.

(b) Find the power spectral density $S_{V'}(z)$ and show that the resulting variance of V' is equal to $\mathbb{E}[V'^2] = (1+\alpha^2+2\alpha\beta+\beta^2+\alpha^2\beta^2)\frac{N_0}{2}$.

(c) To analyze the performance, we group all terms that are not the desired symbol $I[n]$ as follows:

$$U'_{ZF}[n] = f[0]I[n] + \underbrace{\sum_{k=-\infty, k\neq 0}^{k=+\infty} f[k]I[n-k] + V'[n]}_{G[n]}.$$

Unfortunately, it is pretty cumbersome to find the exact error probability for all these interference and correlated noise terms. Instead, we apply a simplification that is common in communication engineering: We aggregate all these undesired terms and pretend that they are just one independent Gaussian term $G[n]$. What are its mean and variance? If we pretend it is Gaussian, find the resulting single bit-error probability in terms of a Q-function. Comment on the error probability, at high SNR regime.