ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 8 Advanced Digital Communications
Homework 3 (*Graded, Due Oct. 17, 2016*) Oct. 10, 2016

PROBLEM 1. (15 pts)

(a) Let $X_1, X_2, \ldots, X_n$ be a sequence of $n > 1$ binary i.i.d. random variables with $\Pr\{X_m = 0\} = \frac{1}{2}$. Let Z be a parity check on $X_1, \ldots, X_n$; i.e., $Z = X_1 \oplus X_2 \oplus \cdots \oplus X_n$.

(i) Is Z independent of X_1?

(ii) Are $Z, X_1, \ldots, X_{n-1}$ independent?

(iii) Are $Z, X_1, \ldots, X_n$ independent?

(iv) Is Z independent of X_1 if $\Pr\{X_m = 0\} \neq \frac{1}{2}$? (You may take $n = 2$ here.)

(b) Let $\boldsymbol{Z} = (Z_1, \ldots, Z_n)$ denote a jointly Gaussian vector with independent components each with zero mean and variance σ^2, i.e., we have

$$f_{\boldsymbol{Z}}(\boldsymbol{z}) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{\|\boldsymbol{z}\|^2}{2\sigma^2}}$$

Let $\{\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n\}$ be any orthonormal basis for $\mathbb{R}^n$ and let $\boldsymbol{W} = (W_1, \ldots, W_n)$ denote a random vector whose components are the projections of $\boldsymbol{Z}$ onto this basis, i.e, $W_i = \langle \boldsymbol{Z}, \boldsymbol{\psi}_i \rangle$. Show that $\boldsymbol{W}$ has the same distribution as $\boldsymbol{Z}$.

PROBLEM 2. (12 pts) Consider the binary hypothesis testing problem with MAP decision. Assume that priors are given by $(\pi_0, 1 - \pi_0)$.

(a) Let $V(\pi_0)$ be the overall probability of error. Write the expression for $V(\pi_0)$.

(b) Show that $V(\pi_0)$ is a concave function of π_0 i.e., for priors $(\pi_0, 1 - \pi_0)$ and $(\pi_0', 1 - \pi_0')$,

$$V(\lambda\pi_0 + (1-\lambda)\pi_0') \geq \lambda V(\pi_0) + (1-\lambda)V(\pi_0'), \qquad \forall \lambda \in [0,1].$$

PROBLEM 3. (14 pts) Consider an arbitrary signal set $A = \{a_j(t) \colon 1 \leq j \leq M\}$. Assume $a_j(t)$ is chosen for transmission with probability p_j. Let $m_A(t) = \sum_j p_j a_j(t)$ be the average signal, and let A' be A translated by $m_A(t)$ so that the average of A' is zero:

$$A' = \{a_j(t) - m_A(t) \colon 1 \leq j \leq M\}.$$

Let $\mathcal{E}_A$ and $\mathcal{E}_{A'}$ denote the average energies of A and A' respectively.

(a) Show that the error probability of an optimum detector for an additive channel is the same for A' as it is for A.

(b) Show that $\mathcal{E}_{A'} = \mathcal{E}_A - \|m_A(t)\|^2$. Conclude that removing the mean m_A is always a good idea.

PROBLEM 4. (14 pts) In this problem we develop further intuition about matched filters. Let

$$Y(t) = x(t) + Z(t)$$

be the channel output, where $Z(t)$ is additive white Gaussian noise of power spectral density $\frac{N_0}{2}$ and $x(t)$ is the transmitted pulse. Let $h(t)$ be an arbitrary pulse, and consider a receiver that passes the received signal through the filter with impulse response $h(t)$ and samples its output at time T to obtain,

$$Y = (h * x)(T) + (h * Z)(T).$$

(a) Compute $\mathbb{E}[(h * Z)(T)]$ and $\text{var}\big((h * Z)(T)\big)$.

(b) Let the signal-to-noise ratio (SNR) be defined as

$$\text{SNR} := \frac{|(h * x)(T)|^2}{\text{var}\big((h * Z)(T)\big)}.$$

Find $h(t)$ that maximizes the SNR. (You may need to use the Cauchy–Schwarz inequality.)

PROBLEM 5. (24 pts) The received signal in a communication system is given by

$$Y(t) = x_i(t) + Z(t) \qquad i = 1, 2,$$

where $Z(t)$ is white Gaussian noise of spectral density $\frac{N_0}{2}$, and $x_1(t)$ and $x_2(t)$ are two different signals of equal energy $\mathcal{E} = \|x_1(t)\|^2 = \|x_2(t)\|^2$, equally probably to be sent.

(a) Let $\varphi_1(t) = \dfrac{x_1(t) - x_2(t)}{\|x_1(t) - x_2(t)\|}$. Find $\varphi_2(t)$ such that $\{\varphi_1(t), \varphi_2(t)\}$ is an orthonormal basis for the space spanned by $\{x_1(t), x_2(t)\}$.

(b) Implement the optimal receiver using only two filters $h_1(t) = \varphi_1(T - t)$ and $h_2(t) = \varphi_2(T - t)$ (for some $T > 0$ to ensure causality).

(c) Simplify the decision rule of the receiver as much as possible. Conclude that the receiver can actually be implemented using only a single filter with impulse response $h_1(t)$.

(d) Compute the error probability of the receiver and deduce that it is minimized when $x_2(t) = -x_1(t)$.

PROBLEM 6. (21 pts) Consider the vector problem

$$\boldsymbol{Y} = \boldsymbol{x} + \boldsymbol{Z}$$

where $\boldsymbol{Z}$ is a vector of jointly Gaussian random variables with zero mean and covariance matrix Σ, which is assumed to be invertible $\boldsymbol{x}$ is uniformly chosen from the set $\{\boldsymbol{x}_0, \boldsymbol{x}_1\}$.

(a) Using eigenvalue decomposition, a positive-definite matrix Σ can be written as $\Sigma = \Phi\Lambda\Phi^H$ where Φ is a unitary matrix and Λ a diagonal matrix. Show that we can also write Σ as $\Sigma = CC^H$ with some C, and express C in terms of Φ and Λ. What is the covariance matrix of the random vector $C^{-1}\boldsymbol{Z}$?

(b) Derive the ML decision rule for $\boldsymbol{x}$ based on the observation $\boldsymbol{Y}$ and compute its error probability.
Hint. First whiten the noise using the result in (a)

(c) Let $\boldsymbol{x}_0 = (1, 0)$ and $\boldsymbol{x}_1 = (0, -1)$. Give the simplest possible decision rule, sketch the decision region in the two-dimensional space of $\boldsymbol{Y}$ for the following two different covariance matrices, and calculate the error probability for each covariance matrix:

$$\Sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \Sigma_2 = \begin{bmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & 1 \end{bmatrix}.$$