ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 6 Advanced Digital Communications
Homework 2 Oct. 3, 2016

PROBLEM 1.

(a) Let $\boldsymbol{Y}$ be a random vector with covariance matrix C (see the definition in Section 2.3 of your lecture notes). Show that the covariance matrix of $\boldsymbol{Z} = A\boldsymbol{Y}$ equals ACA^H.

(b) Let Y_1 and Y_2 be independent Gaussian random variables of mean zero and variance $\sigma^2_{Y_1}$ and $\sigma^2_{Y_2}$, respectively. Show that the random variable $Z_1 = \alpha Y_1 + \beta Y_2$ is also Gaussian (for any real numbers α and β), and find its mean and variance.

(c) Let us also define $Z_2 = \gamma Y_1 + \delta Y_2$ (for any γ and δ). The joint pdf of random variables Z_1 and Z_2 also follows a Gaussian distribution. In particular, we say that Z_1 and Z_2 are *jointly* Gaussian. What is the mean and covariance matrix of their joint pdf? You may use what you did in part (a).

  (i) What happens when $\alpha = \gamma$ and $\beta = \delta$?

  (ii) Under what conditions will Z_1 and Z_2 be independent random variables?

(d) In questions (b) and (c), we saw how we could create correlated Gaussians from independent Gaussians by means of linear combinations. Now let Z_1 and Z_2 just be two arbitrary *correlated* Gaussian random variables of mean zero and any given covariance matrix C. Suppose we now want to work the other way around, i.e., we wish to create independent Gaussians $\begin{bmatrix} Y_1 & Y_2 \end{bmatrix}^T$ from these correlated $\begin{bmatrix} Z_1 & Z_2 \end{bmatrix}^T$. If we again want to use a linear transformation as

$$A \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

what must we choose our matrix A to be? What will be the resulting variances $\sigma^2_{Y_1}$ and $\sigma^2_{Y_2}$?

PROBLEM 2. Consider the binary hypothesis testing problem with two equiprobable hypotheses. Under $\{H = 1\}$, the observable Y has a Laplacian distribution of mean zero and unit variance and under $\{H = 2\}$, Y has a Gaussian distribution of mean zero and unit variance.

(a) Sketch (in one and the same figure) the Laplacian and the Gaussian distributions. It might be wise to use a log-scale for the y-axis.

(b) Show that the ML detector can be expressed as

$$\hat{H}_{ML}(y) = \begin{cases} 1 & \text{if } ||y| - a| \geq b \\ 2 & \text{otherwise} \end{cases}$$

and find a, b.

(c) Determine the resulting error probability. The error probability can be expressed using the Q-function and the $\sinh(\cdot)$ function given by

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx, \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

Problem 3. Consider Gaussian hypothesis testing with arbitrary priors. Prove that in this case, if y_1 and y_2 are elements of the decision region associated to hypothesis i then so is $\alpha y_1 + (1-\alpha) y_2$, where $\alpha \in [0,1]$.

Problem 4.

(a) An observation y may contain information that does not help to determine which message has been transmitted. The irrelevant components may be discarded without the loss of performance. Assume the observation is of the form $y = (y_1, y_2)$. Show that for the ML receiver, if

$$P_{Y_2|Y_1,X}(y_2|y_1,x) = P_{Y_2|Y_1}(y_2|y_1)$$

then y_2 is not needed at the receiver.

(b) Consider the channel that maps the input X to output (Y_1, Y_2, Y_3) as

$$Y_1 = X + N_1 \qquad Y_2 = X + N_2 \qquad Y_3 = X + N_1 + N_2$$

where X, N_1 and N_2 are independent random variables.

(i) Given only y_1, is y_3 irrelevant?

(ii) Given only y_1 and y_2, is y_3 irrelevant?

For the rest of the problem consider the channel that maps the input X to (Y_1, Y_2) as

$$Y_1 = X + N_1 \qquad Y_2 = X + N_2,$$

The transmitted signal X is either -1 or $+1$. The noise random variables N_1 and N_2 are statistically independent of the transmitted signal and each other. Their density functions are

$$P_{N_1}(n) = P_{N_2}(n) = \frac{1}{2}e^{-|n|}$$

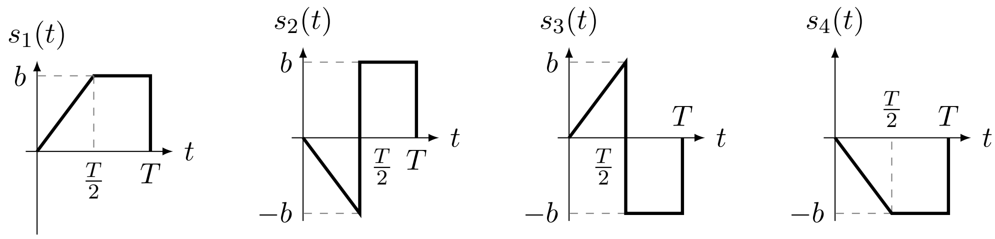
(c) Given y_1, is y_2 irrelevant?

(d) Find the optimum decision regions to minimize the probability of error for equally likely messages.

(e) A receiver decides $X = 1$ if and only if $Y_1 + Y_2 > 0$. Is this receiver optimum for equally likely messages? What is the probability of error?

(f) Find how the optimum decision regions are modified when $\Pr\{X = 1\} > \frac{1}{2}$.

Problem 5. The following set of four waveforms is to be used for transmission across the standard AWGN channel. Assume that $b = \sqrt{6/T}$.



(a) Give an *orthonormal* basis for these waveforms.

(b) Sketch the signal space characterization of this set of waveforms, i.e., the signal points $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{s}_4$.

(c) Determine the energy of each of the waveforms.