ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 4 Advanced Digital Communications
Homework 1 Sep. 26, 2016

Problem 1. Consider a signal $x[n]$, its Fourier transform $X(f)$, and its Z-transform $X(z)$.

(a) Assume $x[n]$ is conjugate-symmetric, that is, $x[n] = x^*[-n]$. Prove that $X(f)$ is real-valued. Furthermore, show that $X(z) = X^*(1/z^*)$.

(b) Consider signals $x[n]$ and $y[n]$ with Fourier transform $X(f)$ and $Y(f)$, respectively. The *convolution* of $x[n]$ and $y[n]$ is defined as

$$(x * y)[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k].$$

Find the Fourier transform of $(x * y)[n]$ in terms of $X(f)$ and $Y(f)$.

(c) Define the signal $\bar{x}[n] = x^*[-n]$. Find the Fourier transform of y defined below. Express the result in terms of $X(f)$. You may use the result from part (b).

$$y[n] = (x * \bar{x})[n] = \sum_{k=-\infty}^{\infty} x[k]\bar{x}[n-k].$$

(d) Show that the discrete Fourier transform can be written in terms of a matrix F that we will refer to as the *Fourier matrix*. Give a general formula for the entries of the Fourier matrix and show that its inverse is simply $F^{-1} = F^H$. Any matrix that satisfies this relationship is called a *unitary* matrix. Finally, write out the Fourier matrix explicitly for dimensions 2, 3, and 4.

Problem 2. Let X and Y be discrete random variables defined on some probability space with a joint pmf $p_{XY}(x, y)$. Let $a, b \in \mathbb{R}$ be constants.

(a) Prove that $\mathbb{E}[aX + bY] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y]$. Do not assume independence.

(b) Prove that if X and Y are independent random variables, then $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

(c) Assume that X and Y are not independent. Find an example where $\mathbb{E}[X \cdot Y] \neq \mathbb{E}[X] \cdot \mathbb{E}[Y]$, and another example where $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

(d) Prove that if X and Y are independent, then they are also uncorrelated, i.e.,

$$\text{cov}(X, Y) := \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] = 0.$$

(e) Assume that X and Y are uncorrelated and let σ_X^2 and σ_Y^2 be the variances of X and Y, respectively. Find the variance of $aX + bY$ and express it in terms of $\sigma_X^2, \sigma_Y^2, a, b$. *Hint.* First show that $\text{cov}(X, Y) = \mathbb{E}[X \cdot Y] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

Problem 3.

(a) For a non-negative integer-valued random variable N, show that

$$\mathbb{E}[N] = \sum_{n>0} \Pr\{N \geq n\}.$$

(b) Show, with whatever mathematical care you feel comfortable with, that for an arbitrary non-negative random variable X,

$$\mathbb{E}[X] = \int_0^\infty \Pr\{X \geq a\}\, da.$$

(c) Derive the Markov inequality, which says that for any non-negative random variable X and positive a, we have

$$\Pr\{X \geq a\} \leq \frac{\mathbb{E}[X]}{a}.$$

Hint. Sketch $\Pr\{X \geq a\}$ as a function of a and compare the area of the rectangle with horizontal length a and vertical length $\Pr\{X \geq a\}$ in your sketch with the area corresponding to $\mathbb{E}[X]$.

(d) Derive the Chebyshev inequality, which says that

$$\Pr\{|Y - \mathbb{E}[Y]| \geq b\} \leq \frac{\sigma_Y^2}{b^2}$$

for any random variable Y with finite mean $\mathbb{E}[Y]$ and finite variance σ_Y^2.

(e) Derive the Chernoff bound, which says that for any random variable Z,

$$\Pr\{Z \geq b\} \leq \mathbb{E}\big[e^{s(Z-b)}\big], \qquad \forall s \geq 0.$$

Problem 4. Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent identically distributed (i.i.d.) random variables with the common probability density function $f_X(x)$. Note that $\Pr\{X_n = \alpha\} = 0$ for all α and that $\Pr\{X_n = X_m\} = 0$, for $m \neq n$.

(a) Find $\Pr\{X_1 \leq X_2\}$. (Give a numerical answer, not an expression; no computation is required and a one- or two-line explanation should be adequate.)

(b) Find $\Pr\{X_1 \leq X_2; X_1 \leq X_3\}$; in other words, find the probability that X_1 is the smallest of $\{X_1, X_2, X_3\}$. (Again, think — do not compute.)

(c) Let the random variable N be the index of the first random variable in the sequence to be less than X_1; i.e.,

$$\{N = n\} = \{X_1 \leq X_2; X_1 \leq X_3; \ldots; X_1 \leq X_{n-1}; X_1 > X_n\}.$$

Find $\Pr\{N \geq n\}$ as a function of n.

(d) Show that $\mathbb{E}[N] = \infty$.

(e) Now assume that $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables each drawn from a finite set of values. Explain why you cannot find $\Pr\{X_1 \leq X_2\}$ without knowing the pmf. Explain $\mathbb{E}[N] = \infty$.