Homework 7 (due Wednesday, November 9)

Exercise 1. Let $d \geq 1$ and $m > 2$ be integers, and let us consider the following process on $S = \{0, \ldots, m-1\}^d$: at each step, from state $x \in S$, pick a component of x uniformly at random and change it to *another* number in $\{0, \ldots, m-1\}$, chosen again uniformly at random.

a) Write down the transition matrix P of this chain. Is this chain is ergodic? What is its stationary distribution? Is the detailed balance equation satisfied?

It turns out that the eigenvectors of P are given by $(\phi^{(z)}, z \in S)$, where

$$\phi_x^{(z)} = \exp(2\pi i x \cdot z/m), \quad x \in S,$$

and $x \cdot z = \sum_{j=1}^d x_j \, z_j$.

b) Compute the corresponding eigenvalues $(\lambda_z, \, z \in S)$ of P.

Hint: Express these in terms of $|z| = \sharp\{$ non-zero components of $z\}$.

c) Deduce the value of the spectral gap for $d > 2$, as well as a corresponding upper bound on $\|P_0^n - \pi\|_{\mathrm{TV}}$.

d) Compare this upper bound to the general lower bound found in class. Do these two bounds match for large m and d?

e) Through a more careful analysis, find a tighter upper bound on $\|P_0^n - \pi\|_{\mathrm{TV}}$ for large m and d.

Exercise 2. Regarding the lazy random walk on $\{0,1\}^d$, we saw in class that $\|P_0^n - \pi\|_{\mathrm{TV}}$ is arbitrarily close to 1 for

$$n = \frac{d+1}{4} \, (\log d - c)$$

and $c > 0$ arbitrarily large. Following the reasoning made in class (but the technique is simpler here!), show that the following distance:

$$\|P_0^n - \pi\|_2 = \left(\sum_{y \in \{0,1\}^d} \left(\frac{p_{0y}(n)}{\sqrt{\pi_y}} - \sqrt{\pi_y} \right)^2 \right)^{1/2}$$

can be made arbitrarily large by taking again $n = \frac{d+1}{4} \, (\log d - c)$ and $c > 0$ arbitrarily large.

NB: The above distance is the ℓ^2-distance between P_0^n and π; it has been shown in class to be an upper bound on $\|P_0^n - \pi\|_{\mathrm{TV}}$ (with an extra factor 2).