SOLUTION 1.

(a) The state diagram and detour flow graph are shown here. The states are labeled as $\langle b_{j-1}, b_{j-2} \rangle$ and the transitions with $b_j/x_{2j-1}, x_{2j}$.

(b) The output to $-1, -1, \ldots, -1$ will be $-1, -1, 1, 1, 1, \ldots, 1$. 
(c) Given the observation \( y = (y_1, \ldots, y_{2n}) \), the ML codeword is given by \( \arg \max_{x \in C} p(y|x) \) where \( C \) represents the set of codewords (i.e., the set of all possible paths on the trellis). Alternately, the ML codeword is given by

\[
\arg \max_{x \in C} \sum_{i=1}^{2n} \log p(y_i|x_i).
\]

Hence, a branch metric for the BSC is

\[
\log p(y_i|x_i) = \begin{cases} 
\log \epsilon & \text{if } y_i \neq x_i, \\
\log(1 - \epsilon) & \text{if } y_i = x_i.
\end{cases}
\]

The decoder chooses the path with the largest metric.

(d) The channel output will be 1, 1, \ldots, 1 (length \( 2n \) all-1 sequence). The decoder will clearly choose the path corresponding to the all-1 input sequence on the trellis and, hence, decode the maximum likelihood transmitted input sequence as 1, 1, \ldots, 1 (length \( n \) all-1 sequence).

**Solution 2.**

(a) Firstly we have

\[
\psi_F(f) = e^{-\pi f^2}(1 - e^{-j\pi f}) = 2j e^{-j\pi f/2} e^{-\pi f^2} \sin \left( \frac{\pi}{2} f \right).
\]

Moreover, \( \mathbb{E}[X_i] = 0 \) and

\[
K_X[k] = \mathbb{E}[X_{i+k}X_{i}^*] = \mathcal{E} \mathbb{1} \{ k = 0 \}.
\]

Thus,

\[
S_X(f) = |\psi_F(f)|^2 \sum_k K_X[k] e^{-j2\pi kf} = 4\mathcal{E} e^{-2\pi f^2} \sin^2 \left( \frac{\pi}{2} f \right).
\]

Therefore, \( S_X(f) = 0 \) for \( \forall f = 2m, \ m \in \mathbb{Z} \).

(b) It is easy to check that still \( \mathbb{E}[X_i] = 0 \) and

\[
K_X[k] = \mathbb{E}[X_{i+k}X_{i}^*] = s^2 (1 + \alpha^2) = \mathcal{E} \implies s = \pm \sqrt{\frac{\mathcal{E}}{1 + \alpha^2}}.
\]

Therefore,

\[
S_X(f) = 4\mathcal{E} \frac{e^{-2\pi f^2} \sin^2 \left( \frac{\pi}{2} f \right)}{1 + \alpha^2} ((1 + \alpha^2) + 2\alpha \cos (4\pi f)\).
\]

Finally, if \( |\alpha| \neq 1, \) since \( (1 + \alpha^2) + 2\alpha \cos (4\pi f) \) has no real zeros for \( f \), \( S_X(f) = 0 \) for \( \forall f = 2m, \ m \in \mathbb{Z} \). However, if \( \alpha = 1, \) solving \( (1 + \alpha^2) + 2\alpha \cos (4\pi f) = 0 \) for \( f \) gives additional zeros at \( f = \frac{2m+1}{4}, \ m \in \mathbb{Z} \). Similarly if \( \alpha = -1 \) there will be additional nulls at frequencies \( f = \frac{m}{2}, \ m \in \mathbb{Z} \).
(c) Since \( d, d' \in \{-1, +1\} \), we can express \( f(d, d') \) as
\[
f(d, d') = \frac{1}{2}(d - d').
\]
Once again we have \( \mathbb{E}[X_i] = 0 \) and

\[
K_X[k] = \mathbb{E}[X_{i+k}X_i^*] = \frac{s^2}{4} (\mathbb{E}[D_{i+k}D_i] - \mathbb{E}[D_{i+k}D_{i-1}] - \mathbb{E}[D_{i+k-1}D_i] + \mathbb{E}[D_{i+k-1}D_{i-1}])
\]
which is
\[
= \frac{s^2}{4} (2 \mathbb{I}\{k = 0\} - 1 \{k = -1\} - 1 \{k = 1\}).
\]
Thus
\[
K_X[0] = \frac{s^2}{2} = \mathcal{E} \implies s = \pm \sqrt{2\mathcal{E}},
\]
and
\[
S_X(f) = 4\mathcal{E}e^{-2\pi f^2} \sin^2 \left( \frac{\pi}{2} f \right) (1 - \cos (2\pi f))
\]

(d) Using the precoder of (c) \( S_X(f) = 0 \) for \( \forall f = m, \ m \in \mathbb{Z} \) (thus, in particular \( S_X(1) = 0 \)).
Using the precoding proposed in (b) we have
\[
S_X(1) = 4\mathcal{E}e^{-2\pi (1 + \alpha)^2} = 0 \iff \alpha = -1.
\]

**Solution 3.**

(a) Based on Nyquist’s criterion we know that \( B \geq \frac{1}{2} \).

(b) If \( B = \frac{1}{2} \), in order for \( \psi(t) \) to be unit-norm and orthogonal to its 1-shifts we must have
\[
|\psi_F(f)|^2 = \mathbb{I}\{-\frac{1}{2} \leq f \leq \frac{1}{2}\}.
\]
Therefore,
\[
\psi_F(f) = e^{-j2\pi ft_0} \mathbb{I}\{-\frac{1}{2} \leq f \leq \frac{1}{2}\} \iff \psi(t) = \text{sinc}(t - t_0)
\]
Finally solving for \( \psi(0) = 0 \) gives \( t_0 = 0 \). Thus \( \psi(t) = \text{sinc}(t) \).

(c) If \( \theta = 0 \),
\[
\mathbb{R}\{R_E(t)\} = \mathbb{R}\{w_E(t)\} + N_R(t),
\]
\[
\mathbb{I}\{R_E(t)\} = \mathbb{I}\{w_E(t)\} + N_I(t),
\]
where \( N_R(t) \) and \( N_I(t) \) are independent white Gaussian noise processes of power spectral density \( \frac{N_0}{2} \).

A sufficient statistic to estimate \( X_j \) from the received signal is obtained by computing the (complex-valued) inner products
\[
Y_j = \langle R_E(t), \psi(t - j) \rangle,
\]
or equivalently, pairs of real-valued inner products
\[
Y_{1,j} = \langle \mathbb{R}\{R_E(t)\}, \psi(t - j) \rangle \quad \text{and} \quad Y_{2,j} = \langle \mathbb{I}\{R_E(t)\}, \psi(t - j) \rangle.
\]
To this end, one in principle has to filter the outputs of the down-converter using matched filters of impulse response $\psi^*(-t)$ and sample the outputs of the filters at times $t = j, j \in \mathbb{Z}$. However, in this problem we see that a filter with impulse response $\psi^*(-t)$ is nothing but a low-pass filter with frequency response $\mathbb{1}\{ -\frac{1}{2} \leq f \leq \frac{1}{2} \}$ which is already included in the down-converter. Thus, it is sufficient to sample the output of the down-converters directly to obtain the desired sufficient statistics.

\[
\sqrt{2} \cos(2\pi f_c t + \theta) \\
R(t) \times \\
\mathbb{1}\{ -B \leq f \leq B \} \\
\mathbb{1}\{ -B \leq f \leq B \} \\
-\sqrt{2} \sin(2\pi f_c t + \theta)
\]

(d) We have the following hypothesis testing problem:

\[
\text{under } H = i : \quad Y = c_i + Z,
\]

where $Z \sim \mathcal{N}(0, \frac{N_0}{2} I_2)$ and $c_1 = [1, 0]$, $c_2 = [0, 1]$, $c_3 = [-1, 0]$, and $c_4 = [0, -1]$.

For an AWGN setting, the ML decision rule will be the minimum distance decision rule with the following decision regions:

This is a 4-PSK constellation and the probability of error of an ML decoder for such a constellation is

\[
P_e = 2Q\left( \frac{1}{\sqrt{N_0}} \right) - Q\left( \frac{1}{\sqrt{N_0}} \right)^2.
\]

(e) Using the trigonometric identity $\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)$ we can see that the output of the top modulator, in presence of the phase difference, is

\[
R(t) \cos(\theta) \times \sqrt{2} \cos(2\pi f_c t) - R(t) \sin(\theta) \times \sqrt{2} \sin(2\pi f_c t).
\]

Thus, as the low-pass filter is a linear system, the output of the top low-pass filter is:

\[
\Re\{R_E(t)\} = \Re\{w_E(t)\} \cos(\theta) + \Im\{w_E(t)\} \sin(\theta) + \cos(\theta) N_R(t) + \sin(\theta) N_I(t).
\]
Similarly, we can show that the output of the bottom low-pass filter is:

$$\Im \{ R_E(t) \} = \Im \{ w(t) \} \cos(\theta) - \Re \{ w(t) \} \sin(\theta) + \cos(\theta) N_I(t) - \sin(\theta) N_R(t).$$

Therefore, the observable $Y = [Y_1, Y_2]$ (under $H = i$) is now equal to

$$Y = R_\theta c_i + R_\theta Z$$

where

$$R_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

is the rotation matrix, the codewords $c_i$ are as in part (d), and $Z \sim \mathcal{N}(0, \frac{N_0}{2} I_2)$. Moreover, we know that $V = R_\theta Z$ has the same statistics as $Z$. Thus, we can write the observable $Y$ as

$$Y = R_\theta c_i + V$$

with $V \sim \mathcal{N}(0, \frac{N_0}{2} I_2)$.

Using the above diagram, we can see that the probability of error of the receiver is

$$P_e = Q \left( \frac{\sin \left( \frac{\pi}{4} - \theta \right)}{\sqrt{N_0/2}} \right) + Q \left( \frac{\cos \left( \frac{\pi}{4} - \theta \right)}{\sqrt{N_0/2}} \right) - Q \left( \frac{\sin \left( \frac{\pi}{4} - \theta \right)}{\sqrt{N_0/2}} \right) - Q \left( \frac{\cos \left( \frac{\pi}{4} - \theta \right)}{\sqrt{N_0/2}} \right).$$

(f) If $|\theta| > \frac{\pi}{4}$, the constellation will be rotated in such a way that each codeword will be moved out of its corresponding decision region (e.g. $c_1$ will be moved to the decision region $\mathcal{R}_2$, $c_2$ to $\mathcal{R}_3$, ...). Therefore, in the absence of noise the decoder always decodes the sent codeword incorrectly (error probability is 1). As the noise variance increases the error probability decreases (there is a higher chance for the noise to move the observable $Y$ into the correct decision region). In particular if the noise variance goes to infinity the observable $Y$ will be a point chosen on the $\mathbb{R}^2$ plane uniformly at random. Thus, with probability $\frac{1}{4}$ it will be in the decoding region corresponding to the transmitted codeword which means the error probability will be decreased to $\frac{3}{4}$. 