Solution 1.

(a) $\sigma_a^2 = 1$; any invertible transform on the output — in particular multiplication by 2 — does not change the error probability.

(b) $\sigma_b^2 = \frac{1}{3}$; $Y = (\pm 2) + W$ which is equivalent to $Y' = \frac{1}{2}Y = (\pm 1) + \frac{1}{2}W$ and $Z = \frac{1}{2}W \sim \mathcal{N}(0, \frac{1}{4})$.

(c) $\sigma_c^2 = 2$; $Y = (\pm 1) + W_1 + W_2$ and $W_1 + W_2 \sim \mathcal{N}(0, 2)$ (since $W_1$ and $W_2$ are independent).

(d) $\sigma_d^2 = 1$; $Y_1$ is a sufficient statistic for decision.

(e) $\sigma_e^2 = \frac{1}{2}$; the observable is

$$(Y_1, Y_2) = (\pm 1, 1) + (W_1, W_2)$$

where $(W_1, W_2) \sim \mathcal{N}(0, I_2)$ and $\frac{1}{2}(Y_1 + Y_2) = \pm 1 + Z$ where $Z = \frac{1}{2}(W_1 + W_2) \sim \mathcal{N}(0, \frac{1}{2})$ is a sufficient statistic for the decision.

Solution 2.

(a) Under the hypothesis $H = +1$, $(Y_1, \ldots, Y_n)$ is an i.i.d. sequence whose components are Laplacian random variables with mean 1, namely

$$f_{Y_1, \ldots, Y_n|H}(y_1, \ldots, y_n|+1) = \left(\frac{1}{2}\right)^n \exp\left\{-\sum_{k=1}^n |y_k - 1|\right\}.$$

Similarly,

$$f_{Y_1, \ldots, Y_n|H}(y_1, \ldots, y_n|-1) = \left(\frac{1}{2}\right)^n \exp\left\{-\sum_{k=1}^n |y_k + 1|\right\}.$$

The MAP decision rule is

$$\frac{f_{Y_1, \ldots, Y_n|H}(y_1, \ldots, y_n|+1)}{f_{Y_1, \ldots, Y_n|H}(y_1, \ldots, y_n|-1)} \stackrel{\hat{H} = +1}{\sim} \frac{1 - p}{p},$$

which, after canceling the common factors and taking the logarithm, becomes

$$\sum_{k=1}^n (|y_k + 1| - |y_k - 1|) \stackrel{\hat{H} = +1}{\geq} \ln \frac{1 - p}{p}. \quad (1)$$

(b) Since $\forall \alpha \in \mathbb{R}: |\alpha + 1| - |\alpha - 1| \in [-2, 2]$, the left-hand-side of (1) lies in $[-2n, 2n]$. Therefore, if

$$2n < \ln \frac{1 - p}{p} \iff p < \frac{1}{1 + e^{2n}},$$

the receiver always chooses $\hat{H} = -1$.

Similarly, if

$$-2n > \ln \frac{1 - p}{p} \iff p > \frac{e^{2n}}{1 + e^{2n}},$$

the decision will always be $\hat{H} = +1$ (regardless of the observation).
(c) \( T(y_1, \ldots, y_n) = \sum_{k=1}^n (|y_k + 1| - |y_k - 1|) \) is the log-likelihood ratio and, hence, is a sufficient statistic. We can prove this using Neyman–Fisher factorization theorem by noting that (for \( a \in \{-1,+1\})
\[
f_{Y|H}(y_1, \ldots, y_n|a) = \left(\frac{1}{2}\right)^n \exp\left\{ -\frac{1}{2} \sum_{k=1}^n (|y_k - 1| + |y_k + 1|) \right\} \\
\times \exp\left\{ -\frac{a}{2} \sum_{k=1}^n (|y_k - 1| - |y_k + 1|) \right\}.
\]

(d) We have
\[
f_{V_1, \ldots, V_n|H}(v_1, \ldots, v_n|1) = \exp\left\{ -\sum_{k=1}^n (v_k - 1) \right\} \prod_{k=1}^n 1\{v_k \geq 1\},
\]
and
\[
f_{V_1, \ldots, V_n|H}(v_1, \ldots, v_n|-1) = \exp\left\{ -\sum_{k=1}^n (v_k + 1) \right\} \prod_{k=1}^n 1\{v_k \geq -1\},
\]

Simplifying the above we get (for \( a \in \{-1,+1\})
\[
f_{V_1, \ldots, V_n|H}(v_1, \ldots, v_n|a) = \exp\left\{ -\sum_{k=1}^n v_k \right\} \times \exp(na)1\{\min\{v_1, \ldots, v_n\} \geq a\},
\]
with \( T'(v_1, \ldots, v_n) = \min\{v_1, \ldots, v_n\}\).

Since conditioned on \( H = a \), \( a \in \{-1,+1\} \) the observables \( Y_1, \ldots, Y_n \) and \( V_1, \ldots, V_n \) are independent,
\[
f_{Y_1, \ldots, Y_n, V_1, \ldots, V_n|H}(y_1, \ldots, y_n, v_1, \ldots, v_n|a) = f_{Y_1, \ldots, Y_n|H}(y_1, \ldots, y_n|a) \times f_{V_1, \ldots, V_n|H}(v_1, \ldots, v_n|a)
\]
\[
= h(y_1, \ldots, y_n)h'(v_1, \ldots, v_n) \times g_a(T(y_1, \ldots, y_n))g'_a(T'(v_1, \ldots, v_n))
\]
where \( h, g_a, h', \) and \( g'_a \) are defined in (2) and (3). Therefore, using the factorization theorem we conclude that \( (T(y_1, \ldots, y_n), T'(v_1, \ldots, v_n)) \) is a sufficient statistic for the hypothesis testing problem.

The MAP decision rule (in terms of \( T \) and \( T' \)) is
\[
g_{+1}(T)g'_{+1}(T') \times p \underset{H=+1}{\geq} g_{-1}(T)g'_{-1}(T') \times (1 - p).
\]

Now if \( T' = \min\{v_1, \ldots, v_n\} \in (-1, 1) \) we see that \( g'_{+1}(T') = 0 \) thus the MAP rule always chooses \( H = -1 \). Otherwise (i.e., when \( \min\{v_1, \ldots, v_n\} \geq 1 \)) \( (4) \) reduces to
\[
T(y_1, \ldots, y_n) = \sum_{k=1}^n (|y_k + 1| - |y_k - 1|) \underset{H=+1}{\geq} \underset{H=-1}{\geq} \ln \frac{1 - p}{p} - 2n.
\]
Thus, the decision regions are:
(note that $T \in [-2n, 2n]$ as we discussed in (b) and $T' \geq -1$).

(e) From the decision regions of (d) it is clear that if $p \geq \frac{1}{2}$ the optimal decision depends only on $T'$ which, in turn, is only a function of $(V_1, \ldots, V_n)$. Therefore, if $p \geq \frac{1}{2}$ the receiver that only observes $(V_1, \ldots, V_n)$ can perform as well as the optimal receiver.

**Solution 3.**

(a) Since the space spanned by $\{w_0, w_1\}$ is the same as the space spanned by $\{v_0, w_1\}$, we can obtain $v_1$ by applying the Gram–Schmidt procedure on $\{v_0, w_1\}$:

$$w_1 - \langle w_1, v_0 \rangle v_0 = w_1 - \left\langle w_1, \frac{w_0 - w_1}{\|w_0 - w_1\|} \right\rangle \frac{w_0 - w_1}{\|w_0 - w_1\|}$$

$$= w_1 - \frac{\langle w_0, w_1 \rangle - \|w_1\|^2}{\|w_0 - w_1\|^2} \cdot (w_0 - w_1)$$

$$= w_1 - \frac{\langle w_0, w_1 \rangle - \|w_1\|^2}{\|w_0\|^2 + \|w_1\|^2 - 2\langle w_0, w_1 \rangle} \cdot (w_0 - w_1)$$

$$= w_1 - \frac{\langle w_0, w_1 \rangle - \mathcal{E}}{2\mathcal{E} - 2\langle w_0, w_1 \rangle} \cdot (w_0 - w_1)$$

$$= w_1 + \frac{1}{2} (w_0 - w_1) = \frac{1}{2} (w_0 + w_1).$$

Therefore,

$$v_1 = \frac{w_1 - \langle w_1, v_0 \rangle v_0}{\|w_1 - \langle w_1, v_0 \rangle v_0\|} = \frac{w_0 + w_1}{\|w_0 + w_1\|}.$$  

(b) Let $Z_0 = \langle N, v_0 \rangle$ and $Z_1 = \langle N, v_1 \rangle$. $Z_0$ and $Z_1$ are independent because $v_0$ and $v_1$ are orthogonal. We have:

$$U_1 = \langle R, v_1 \rangle = \begin{cases} 
\langle w_0, \frac{w_0 + w_1}{\|w_0 + w_1\|} \rangle + Z_1 & \text{if 0 is sent,} \\
\langle w_1, \frac{w_0 + w_1}{\|w_0 + w_1\|} \rangle + Z_1 & \text{if 1 is sent.}
\end{cases}$$

$$= \left\langle \frac{\|w_0\|^2 + (w_0, w_1)}{\|w_0 + w_1\|^2} w_0 + w_1, \frac{\|w_0\|^2 + (w_0, w_1)}{\|w_0 + w_1\|^2} w_0 + w_1 \right\rangle + Z_1$$

$$= \left\langle \frac{\mathcal{E} + (w_0, w_1)}{\|w_0 + w_1\|^2} + Z_1, \frac{\mathcal{E} + (w_0, w_1)}{\|w_0 + w_1\|^2} + Z_1 \right\rangle$$

This shows that the distribution of $U_1$ is independent from the transmitted bit (and from $U_0$). Therefore, $U_1$ can be thrown away. Hence, $U_0$ is sufficient statistics for the hypothesis testing problem.
(c) We have:
\[
U_0 = \langle R, v_0 \rangle = \begin{cases} 
\langle w_0, \frac{w_0 - w_1}{\|w_0 - w_1\|} \rangle + Z_0 & \text{if 0 is sent}, \\
\langle w_1, \frac{w_0 - w_1}{\|w_0 - w_1\|} \rangle + Z_0 & \text{if 1 is sent},
\end{cases}
\]
\[= \begin{cases} 
\frac{\|w_0 - w_1\|^2}{2\|w_0 - w_1\|} + Z_0 & \text{if 0 is sent}, \\
-\frac{\|w_0 - w_1\|^2}{2\|w_0 - w_1\|} + Z_0 & \text{if 1 is sent}.
\end{cases}
\]
Note that \(\|w_0 - w_1\|^2 = \|w_0\|^2 + \|w_1\|^2 - 2\langle w_0, w_1 \rangle = 2\mathcal{E} - 2\langle w_0, w_1 \rangle\). Therefore,
\[
U_0 = \begin{cases} 
\frac{1}{2}\|w_0 - w_1\| + Z_0 & \text{if 0 is sent}, \\
-\frac{1}{2}\|w_0 - w_1\| + Z_0 & \text{if 1 is sent}.
\end{cases}
\]
Now since \(Z_0 = \langle N, v_0 \rangle \sim \mathcal{N}(0, N_0^2)\), the probability of error of the MAP decoder is given by
\[
P_e = Q\left(\frac{\frac{1}{2}\|w_0 - w_1\|}{\sqrt{N_0^2 + \mathcal{E}}} \right) = Q\left(\frac{\|w_0 - w_1\|}{\sqrt{2N_0^2}} \right).
\]
(d) The Cauchy–Schwarz inequality gives \(|\langle w_0, w_1 \rangle| \leq \|w_0\| \cdot \|w_1\| = \mathcal{E}\). Therefore, \(\langle w_0, w_1 \rangle \geq -\mathcal{E}\). Hence,
\[
\|w_0 - w_1\|^2 = 2\mathcal{E} - 2\langle w_0, w_1 \rangle \leq 2\mathcal{E} + 2\mathcal{E} = 4\mathcal{E}.
\]
We conclude that \(\|w_0 - w_1\| \leq 2\sqrt{\mathcal{E}}\). Therefore, the probability of error of the MAP decoder is lower-bounded as follows:
\[
P_e = Q\left(\frac{\|w_0 - w_1\|}{\sqrt{2N_0^2}} \right) \geq Q\left(\frac{2\sqrt{\mathcal{E}}}{\sqrt{2N_0^2}} \right) = Q\left(\frac{\sqrt{2\mathcal{E}}}{N_0} \right).
\]
Moreover, \((*)\) becomes an equality when \(\langle w_0, w_1 \rangle = -\mathcal{E} = -\|w_0\| \cdot \|w_1\|\), which is true if \(w_1 = -w_0\).

**Solution 4.**

(a) Looking at the waveforms we realize that the four signals \(\psi_1(t) = 1\{0 \leq t \leq 1\}, \psi_2(t) = \psi_1(t - 1), \psi_3(t) = \psi_1(t - 2),\) and \(\psi_4(t) = \psi_1(t - 2)\) form an orthonormal basis for the signal space spanned by the waveforms. In this basis \(w_1(t), w_2(t), w_3(t),\) and \(w_4(t)\) correspond to the codewords \(c_1 = (2, 1, 3, 2), c_2 = (1, 0, 2, 1), c_3 = (0, -1, 1, 0)\) and \(c_4 = (-1, -2, 0, -1)\) respectively.

An ML receiver (which is optimal because of equiprobable hypotheses) first projects the received signal \(R(t) = w_i(t) + N(t)\) onto the orthonormal basis and forms the 4-tuple \((Y_1, Y_2, Y_3, Y_4)\) with \(Y_k = \langle R(t), \psi_k(t) \rangle, k = 1, 2, 3, 4\). This reduces the problem to the hypothesis testing problem in discrete additive white Gaussian noise channel,

\[
\text{under } H = i, i = 1, 2, 3, 4: \quad Y = c_i + Z
\]
where $c_i$'s are defined above and $Z \sim \mathcal{N}(0, \frac{N_0}{2} I_4)$. We know that the ML receiver should choose $\hat{H} = \arg \min_i \|Y - c_i\|$.

We finally realize that since $h(t) = \psi_1(1 - t)$ and the remaining basis vectors are the shifted versions of $\psi_1(t)$, the $n$-tuple former can be implemented by sampling the output of a single filter at times $t = 1, 2, 3$ and 4 to compute $Y_1, Y_2, Y_3, \text{and } Y_4$ respectively:

![Diagram](image.png)

(b) The union bound gives

$$\Pr\{\text{error}|w_i \text{ is sent}\} \leq \sum_{j \neq i} Q\left(\frac{d_{i,j}}{\sqrt{2N_0}}\right)$$

where $d_{i,j} = \|w_i - w_j\| = \|c_i - c_j\|$. In the following table we have computed those values

<table>
<thead>
<tr>
<th>$d_{i,j}$</th>
<th>1 2 3 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0 2 4 6</td>
</tr>
<tr>
<td>2</td>
<td>2 0 2 4</td>
</tr>
<tr>
<td>3</td>
<td>4 2 0 2</td>
</tr>
<tr>
<td>4</td>
<td>6 4 2 0</td>
</tr>
</tbody>
</table>

Consequently,

$$\Pr\{\text{error}|w_1 \text{ is sent}\} = \Pr\{\text{error}|w_4 \text{ is sent}\} = Q\left(\frac{2}{\sqrt{2N_0}}\right) + Q\left(\frac{4}{\sqrt{2N_0}}\right) + Q\left(\frac{6}{\sqrt{2N_0}}\right),$$

and

$$\Pr\{\text{error}|w_2 \text{ is sent}\} = \Pr\{\text{error}|w_3 \text{ is sent}\} = 2Q\left(\frac{2}{\sqrt{2N_0}}\right) + Q\left(\frac{4}{\sqrt{2N_0}}\right).$$

Therefore,

$$\Pr\{\text{error}\} = \sum_{i=1}^{4} \Pr\{w_i \text{ is sent}\} \Pr\{\text{error}|w_i \text{ is sent}\} \leq \frac{3}{2} Q\left(\frac{2}{\sqrt{2N_0}}\right) + Q\left(\frac{4}{\sqrt{2N_0}}\right) + \frac{1}{2} Q\left(\frac{6}{\sqrt{2N_0}}\right).$$

(c) The minimum energy signal set is obtained by subtracting from each signal the average $\frac{1}{4}[w_1(t) + w_2(t) + w_3(t) + w_4(t)]$ which is depicted below
\[ \frac{1}{4}[w_1(t) + w_2(t) + w_3(t) + w_4(t)] \]

Therefore the minimum energy signal set is

\[ \tilde{w}_1(t) \]
\[ \tilde{w}_2(t) \]
\[ \tilde{w}_3(t) \]
\[ \tilde{w}_4(t) \]

(d) It is easy to verify that the new signal set spans a one-dimensional space with basis \( \tilde{\psi}(t) = \frac{1}{2} \mathbb{1}\{0 \leq t \leq 4\} \). Indeed, the new signal set corresponds to 4-PAM constellation

For the 4-PAM constellation,

\[
\Pr\{\text{error}|w_1 \text{ is sent}\} = \Pr\{\text{error}|w_4 \text{ is sent}\} = Q\left(\frac{2}{\sqrt{2N_0}}\right),
\]

and

\[
\Pr\{\text{error}|w_2 \text{ is sent}\} = \Pr\{\text{error}|w_3 \text{ is sent}\} = 2Q\left(\frac{2}{\sqrt{2N_0}}\right),
\]

which yields

\[
\Pr\{\text{error}\} = \frac{3}{2}Q\left(\frac{2}{\sqrt{2N_0}}\right).
\]

(e) Since translation is an isometric transform and does not change the probability of error, the probability of error for the receiver in part (a) will also be equal to \( \frac{3}{2}Q\left(\frac{2}{\sqrt{2N_0}}\right) \).