PROBLEM 1. (15 points) Consider the rate $\frac{1}{2}$ convolutional code described by equations

$$x_{2j-1} = b_j b_{j-1}, \qquad x_{2j} = b_j b_{j-2},$$

where $b_1, \ldots, b_n \in \{\pm 1\}$ are the inputs to the encoder and $x_1, \ldots, x_{2n}$ are its output bits.

(a) (4 pts) Draw the state diagram and the detour flow graph for the encoder.

(b) (3 pts) Suppose the length-n sequence $-1, -1, \ldots, -1$ is the input to the encoder. What is the output of the encoder (the initial state of the encoder is $(b_0, b_{-1}) = (1, 1)$)?

The output of the encoder is sent through a Binary Symmetric Channel, described by

$$P_{Y|X}(1|1) = P_{Y|X}(-1|-1) = 1 - \epsilon, \quad P_{Y|X}(-1|1) = P_{Y|X}(1|-1) = \epsilon.$$

(c) (4 pts) Derive the branch metric for a maximum likelihood Viterbi decoder and specify whether the decoder chooses the path with largest or smallest path metric.

(d) (4 pts) Suppose that during the transmission of the encoded sequence you found in part (b) the channel introduces 3 errors, *flipping the first, the second, and the fourth bits*. What is the channel output? Upon observing this channel output, what would the Viterbi decoder estimate as the maximum likelihood transmitted information sequence?

PROBLEM 2. (15 points) Suppose the random process $X(t)$ is generated as

$$X(t) = \sum_i X_i \psi(t - i - \Theta)$$

where $\{X_i\}_{i \in \mathbb{Z}}$ is the input to the waveform former, Θ is uniformly distributed in $[0, 1]$ and $\psi(t)$ is given by

$$\psi(t) = e^{-\pi t^2} - e^{-\pi (t-1/2)^2}.$$

(a) (3 pts) Suppose $X_i = \sqrt{\mathcal{E}} D_i$ where $\{D_i\}_{i \in \mathbb{Z}}$ are i.i.d. random variables taking values in $\{+1, -1\}$ with equal probability. Find $S_X(f)$, the power spectrum of $X(t)$. At what frequencies does $S_X(f)$ equal zero?

Hint: $e^{-\pi t^2}$ and $e^{-\pi f^2}$ are Fourier transform pairs.

(b) (4 pts) Consider now the following "precoding" where we change how the data symbols $\{D_i\}_{i \in \mathbb{Z}}$ are transformed to $\{X_i\}_{i \in \mathbb{Z}}$:

$$X_i = s[D_i + \alpha D_{i-2}],$$

where s is a constant scaling factor. How should we choose s to make sure that $\mathbb{E}[X_i^2] = \mathcal{E}$? Redo (a) with this choice of X_i.

(c) (4 pts) Redo (b) if we replace the precoder with

$$X_i = s f(D_i, D_{i-1}) \quad \text{where} \quad f(d, d') = d \, \mathbb{1}\{d \neq d'\}.$$

(d) (4 pts) Suppose we wish to ensure $S_X(1) = 0$. Can we do so with the precoder in (c)? Can we do so with the precoder in (b) with an appropriate choice of α? Explain your answers and find α if it exists.

PROBLEM 3. (20 points) In a 4-PSK passband communication system, given the i.i.d. data sequence $X_j \in \{+1, -1, +j, -j\}$, the transmitter creates the complex-valued baseband waveform $w_E(t)$ as

$$w_E(t) = \sum_j X_j \psi(t - j),$$

where $\psi(t)$ is an arbitrary unit-norm pulse orthogonal to its unit-time shifts with bandwidth B, i.e., $\psi_{\mathcal{F}}(f) = 0$ for $|f| > B$. The baseband signal $w_E(t)$ is, subsequently, up-converted to a passband signal $w(t)$ around the carrier frequency $f_c \gg B$ using the following up-converter:
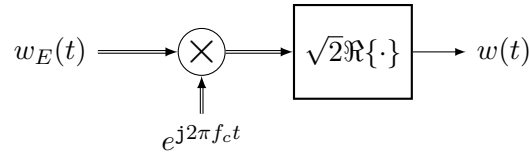
The receiver *which is only capable of performing real-valued operations*, observes $R(t)$, the noisy version of $w(t)$ after passing through a continuous-time AWGN channel of noise power spectral density $\frac{N_0}{2}$, and down-converts it to baseband. However, there is a phase difference of θ between the oscillators of the transmitter and the receiver:
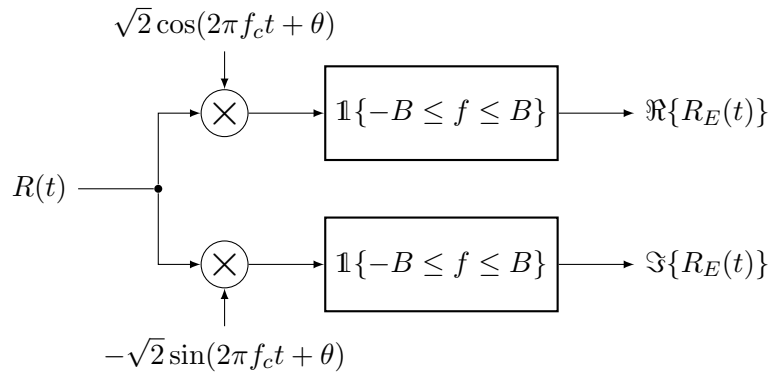
(a) (2 pts) What is the minimum required bandwidth B in order for $\psi(t)$ to be orthogonal to its unit-time shifts?

(b) (3 pts) Determine the pulse shape $\psi(t)$ having the minimal bandwidth you found in part (a) and satisfying $\psi(0) = 1$.

(c) (4 pts) Assuming $\psi(t)$ is the pulse you found in part (b) *and* $\theta = 0$, complete the block-diagram of the receiver to form a two-dimensional sufficient statistic for estimating the data symbols X_j. Do you need any additional filters?

(d) (4 pts) Determine the ML decision rule (for estimating the data symbols X_j), sketch the decision regions, and express the probability of error of the receiver in terms of Q functions (still assuming $\theta = 0$).

(e) (4 pts) Now assume $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4})$ and is unknown to the receiver. Express the probability of error of the receiver of part (d) as a function of θ.

(f) (3 pts) Explain qualitatively what will happen if $|\theta| > \frac{\pi}{4}$?

2