SOLUTION 1.

(a) Notice that

$$\|w_0(t)\|^2 = \|w_1(t)\|^2 = \int_0^{2T} w_0^2(t)dt = 2T$$

We first apply the Gram–Schmidt algorithm. We get the first basis vector from the first signal:

$$\psi_0(t) = \frac{w_0(t)}{\|w_0(t)\|} = \begin{cases} \frac{1}{\sqrt{2T}} & t \in [0, 2T] \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $\psi_0(t)$ and $w_1(t)$ are orthogonal. Thus we obtain the second basis vector by normalizing $w_1(t)$:

$$\psi_1(t) = \frac{w_1(t)}{\|w_1(t)\|} = \begin{cases} \frac{1}{\sqrt{2T}} & t \in [0, T] \\ -\frac{1}{\sqrt{2T}} & t \in [T, 2T] \\ 0 & \text{otherwise} \end{cases}$$

In the $\{\psi_0(t), \psi_1(t)\}$ basis, it is straightforward to see that $c_0 = (\sqrt{2T}, 0)^\mathsf{T}$ and $c_1 = (0, \sqrt{2T})^\mathsf{T}$.
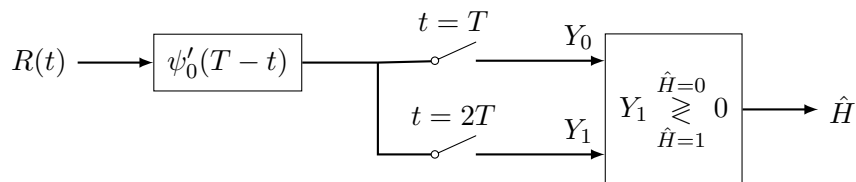
The other basis is the following:

$$\psi_0'(t) = \begin{cases} \frac{1}{\sqrt{T}} & t \in [0, T] \\ 0 & \text{otherwise} \end{cases} \qquad \psi_1'(t) = \begin{cases} \frac{1}{\sqrt{T}} & t \in [T, 2T] \\ 0 & \text{otherwise} \end{cases}$$

Observe that $\psi_1'(t) = \psi_0'(t - T)$. Hence, one matched filter at the receiver sampled twice suffices to project the received signal onto $\psi_0'(t)$ and $\psi_1'(t)$.

In the $\{\psi_0'(t), \psi_1'(t)\}$ basis, the codewords are $c_0 = (\sqrt{T}, \sqrt{T})^\mathsf{T}$ and $c_1 = (\sqrt{T}, -\sqrt{T})^\mathsf{T}$.

(b) The ML receiver is shown below.

Notice that Y_0 is not used. This is not surprising when we look at the signals: For $t \in [0, T]$, the two signals are identical.

(c) We calculate

$$\|w_0(t) - w_1(t)\| = 2\sqrt{T},$$

hence

$$P_e = Q\left(\frac{\sqrt{T}}{\sqrt{N_0/2}}\right)$$

(d) Translating the signal points by any vector will not influence the error probability. However, if the translation vector is the center of mass of the original signal constellation, then the resulting signals will have minimum energy. We compute $v(t) = \frac{1}{2}w_0(t) + \frac{1}{2}w_1(t)$, thus

$$\tilde{w}_0(t) = w_0(t) - v(t) = \begin{cases} 1 & \text{for } t \in [T, 2T] \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{w}_1(t) = w_1(t) - v(t) = \begin{cases} -1 & \text{for } t \in [T, 2T] \\ 0 & \text{otherwise} \end{cases}$$

The resulting signal waveforms are shown below:
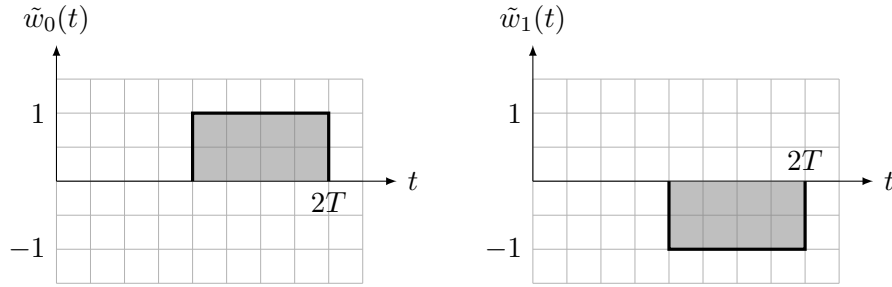
(e) The new signal constellation is antipodal. One can see that

$$\tilde{w}_0(t) = w_0(t) - v(t) = \frac{1}{2}w_0(t) - \frac{1}{2}w_1(t)$$

$$\tilde{w}_1(t) = w_1(t) - v(t) = \frac{1}{2}w_1(t) - \frac{1}{2}w_0(t) = -\tilde{w}_0(t)$$

This shows that we obtain an antipodal signal constellation regardless of the initial waveforms.

SOLUTION 2.

(a) We first compute the centroid of the signal set:

$$a(t) = \sum_{j=0}^{m-1} P_H(j)w_j(t) = \frac{1}{m}\sum_{j=0}^{m-1} w_j(t)$$

The minimum-energy signal set is then obtained by translation:

$$\tilde{w}_j(t) = w_j(t) - a(t) = w_j(t) - \frac{1}{m}\sum_{i=0}^{m-1} w_i(t)$$

$$= \frac{m-1}{m}w_j(t) - \frac{1}{m}\sum_{i\neq j} w_i(t)$$

(b)

$$\|\tilde{w}_j(t)\|^2 = \langle \tilde{w}_j(t), \tilde{w}_j(t) \rangle$$

$$= \langle \frac{m-1}{m} w_j(t) - \frac{1}{m} \sum_{i \neq j} w_i(t), \frac{m-1}{m} w_j(t) - \frac{1}{m} \sum_{k \neq j} w_k(t) \rangle$$

$$= \left(\frac{m-1}{m} \right)^2 \mathcal{E} + \frac{1}{m^2} \sum_{i \neq j} \sum_{k \neq j} \langle w_i(t), w_k(t) \rangle$$

$$= \left(\frac{m-1}{m} \right)^2 \mathcal{E} + \frac{m-1}{m^2} \mathcal{E} = \left(1 - \frac{1}{m} \right) \mathcal{E},$$

and since all signals in $\tilde{\mathcal{W}}$ are equiprobable, we obtain $\tilde{\mathcal{E}} = \left(1 - \frac{1}{m} \right) \mathcal{E}$. The energy saving is therefore $\mathcal{E} - \tilde{\mathcal{E}} = \frac{1}{m} \mathcal{E}$. Alternatively, we could use $\mathcal{E} - \tilde{\mathcal{E}} = \|a(t)\|^2 = \frac{1}{m} \mathcal{E}$.

(c) Notice that $\sum_{j=0}^{m-1} \tilde{w}_j(t) = 0$ by the definition of $\tilde{w}_j(t)$, $j = 0, 1, \ldots, m-1$. Hence the m signals $\{\tilde{w}_0(t), \ldots, \tilde{w}_{m-1}(t)\}$ are linearly dependent. This means that their space has dimensionality less than m. We show that any collection of $m-1$ or less is linearly independent. That would prove that the dimensionality of the space $\{\tilde{w}_0(t), \ldots, \tilde{w}_{m-1}(t)\}$ is $m-1$. Without loss of essential generality we consider $\tilde{w}_0(t), \ldots, \tilde{w}_{m-2}(t)$. Assume that $\sum_{j=0}^{m-2} \alpha_j \tilde{w}_j(t) = 0$. Using the definition of $\tilde{w}_j(t)$, we may write

$$\sum_{j=0}^{m-2} \alpha_j \left(w_j(t) - \frac{1}{m} \sum_{i=0}^{m-1} w_i(t) \right) = 0,$$

$$\left(\sum_{j=0}^{m-2} \alpha_j w_j(t) \right) - \left(\frac{1}{m} \sum_{j=0}^{m-2} \alpha_j \right) \sum_{i=0}^{m-1} w_i(t) = 0,$$

$$\left(\sum_{j=0}^{m-2} \alpha_j w_j(t) \right) - \left(\beta \sum_{i=0}^{m-1} w_i(t) \right) = 0,$$

where $\beta = \frac{1}{m} \sum_{j=0}^{m-2} \alpha_j$. Therefore,

$$\sum_{j=0}^{m-2} (\alpha_j - \beta) w_j(t) - \beta w_{m-1}(t) = 0.$$

But $w_0(t), w_1(t), \ldots, w_{m-1}(t)$ is an orthogonal set and this implies $\beta = 0$ and $\alpha_j = \beta = 0$, $j = 0, 1, \ldots, m-2$. Hence $\tilde{w}_j(t)$, $j = 0, 1, \ldots, m-2$ are linearly independent. We have proved that the new set spans a space of dimension $m-1$.

SOLUTION 3.

(a) In this basis the signal representations are $c_1 = (2, 0, 0, 2)^\mathsf{T}$, $c_2 = (0, 2, 2, 0)^\mathsf{T}$, $c_3 = (2, 0, 2, 0)^\mathsf{T}$, $c_4 = (0, 2, 0, 2)^\mathsf{T}$.

(b) The union bound is expressed in terms of the pairwise distances d_{ij} between the signals since
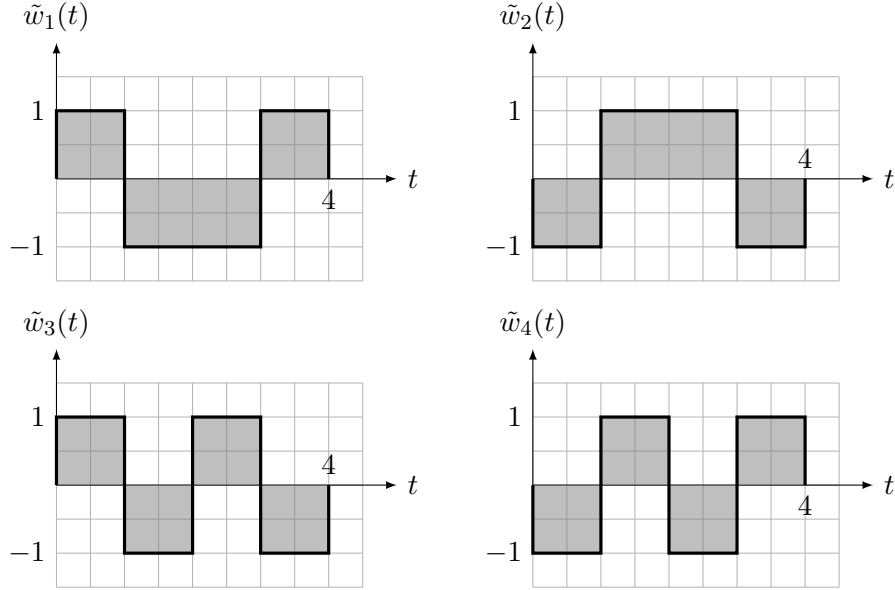
$$P_e(i) \leq \sum_{j \neq i} Q \left(\frac{d_{ij}}{2\sigma} \right)$$

3

From (a) we observe that $d_{12}^2 = d_{34}^2 = 16$ and $d_{13}^2 = d_{14}^2 = d_{23}^2 = d_{24}^2 = 8$, hence

$$P_e(i) \leq 2Q\left(\frac{2}{\sqrt{N_0}}\right) + Q\left(\frac{2\sqrt{2}}{\sqrt{N_0}}\right)$$

Since $P_e(i)$ does not depend on i, it also bounds the average error probability.

(c) The minimum-energy signal set is obtained by subtracting from $\{w_i(t)\}_{i=1}^4$ the average signal $a(t) = \frac{1}{4}\sum_{i=1}^4 w_i(t) = \mathbb{1}_{[0,4]}(t)$. The resulting signals are shown below.

(d) Note that in the new signal set $\tilde{w}_2(t) = -\tilde{w}_1(t)$ and $\tilde{w}_4(t) = -\tilde{w}_3(t)$. Furthermore the signals $\tilde{w}_1(t)$ and $\tilde{w}_3(t)$ are orthogonal. Thus the new signal space is two-dimensional, and the Gram–Schmidt procedure will produce the orthonormal basis $\tilde{\psi}_1(t) = \frac{\tilde{w}_1(t)}{\|\tilde{w}_1\|} = \frac{1}{2}\tilde{w}_1(t)$ and $\tilde{\psi}_2(t) = \frac{\tilde{w}_3(t)}{\|\tilde{w}_3\|} = \frac{1}{2}\tilde{w}_3(t)$.

(e) In the new basis the signal representations are $\tilde{c}_1 = (2,0)^\mathsf{T}$, $\tilde{c}_2 = (-2,0)^\mathsf{T}$, $\tilde{c}_3 = (0,2)^\mathsf{T}$, $\tilde{c}_4 = (0,-2)^\mathsf{T}$. These codewords correspond to those of the 4-QAM constellation (rotated by 45 degrees). The error probability of this set is

$$P_e = 1 - \left[1 - Q\left(\frac{2}{\sqrt{N_0}}\right)\right]^2 = 2Q\left(\frac{2}{\sqrt{N_0}}\right) - Q\left(\frac{2}{\sqrt{N_0}}\right)^2$$

(f) Since translations of a signal set do not change the probability of error, the error probability of the receiver in (b) is equal to that in (e).

4