**Problem 1.** Note that $E_0 = E_1 \cup E_2 \cup E_3$.

(a) (1) For disjoint events, $P(E_0) = P(E_1) + P(E_2) + P(E_3)$, so $P(E_0) = \frac{3}{4}$.

(2) For independent events, $1 - P(E_0)$ is the probability that none of the events occur, which is the product of the probabilities that each one doesn’t occur. Thus $1 - P(E_0) = \left(\frac{3}{4}\right)^3$ and $P(E_0) = \frac{37}{64}$.

(3) If $E_1 = E_2 = E_3$, then $E_0 = E_1$ and $P(E_0) = \frac{1}{4}$.

(b) (1) From the following Venn diagram, $P(E_0)$ is clearly maximized when the events are disjoint, so $\max P(E_0) = \frac{3}{4}$.

(2) The intersection of each pair of sets has probability $\frac{1}{16}$. As seen in the Venn diagram below, $P(E_0)$ is maximized if all these pairwise intersections are identical, in which case $P(E_0) = 3(1/4 - 1/16) + 1/16 = 5/8$. One can also use the formula $P(E_0) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$, and notice that all the terms except the last is fixed by the problem, and the last term cannot be made more than $\min_{i,j} P(E_i \cap E_j) = \frac{1}{16}$.

(c) Same considerations as in (b)(2) yields the upper bound $P(E_0) \leq 3p - 2p^2$ As $P(E_0) = 1$, we find that $p \geq \frac{1}{2}$.

**Problem 2.** Let $L$ be the event that the loaded die is picked and $H$ the event that the honest die is picked. Let $A_i$ be the event that $i$ is turned up on the first roll, and $B_i$ be the event that $i$ is turned up on the second roll. We are given that $P(L) = 1/3; P(H) = 2/3; P(A_i | L) = 2/3; P(A_i | L) = 1/15 \quad 2 \leq i \leq 6; P(A_i | H) = 1/6 \quad 1 \leq i \leq 6$. Then

$$P(L | A_1) = \frac{P(L, A_1)}{P(A_1)} = \frac{P(A_1 | L) P(L)}{P(A_1 | L) P(L) + P(A_1 | H) P(H)} = \frac{2}{3}.$$
This is the probability that the loaded die was picked conditional on the first roll showing a 1. For two rolls we make the assumption from the physical mechanism involved in rolling a die that the outcome on the two successive rolls of a given die are independent. Thus \( P(A_1B_1 \mid L) = \frac{2}{3} \) and \( P(A_1B_1 \mid H) = \frac{1}{6} \). It follows as before that

\[
P(L \mid A_1B_1) = \frac{8}{9}.
\]

**Problem 3.** Since \( A, B, C, D \) form a Markov chain their probability distribution is given by

\[
p(a)p(b|a)p(c|b)p(d|c)
\]

(a) Yes: Summing (1) over \( d \) shows that \( A, B, C \) have the probability distribution

\[
p(a)p(b|a).
\]

(b) Yes: The reverse of a Markov chain is also a Markov chain. Applying this to \( A, B, C, D \) and using part (a) we get that \( D, C, B \) is a Markov chain. Reversing again we get the desired result.

Alternatively summing (1) over \( a \) (noting that \( \sum a p(a) p(b|a) = p(b) \)) shows \( B \xrightarrow{\leftarrow} C \xrightarrow{\leftarrow} D \).

(c) Yes: Since \( A, B, C, D \) is a Markov chain, given \( C, D \) is independent of \( B \), and thus \( p(d|c) = p(d|(b,c)) \). So (1) can be written as

\[
p(a, (b,c), d) = p(a)p((b,c)|a)p(d|(b,c)).
\]

(d) Yes, by a similar (in fact easier) reasoning as (c).

**Problem 4.** No. Take for example \( A = D \) and let \( A \) be independent of the pair \( (B, C) \). Then both \( A, B, C \) and \( B, C, A \) (same as \( B, C, D \)) are Markov chains. But \( A, B, C, D \) is not: \( A \) is not independent of \( D \) when \( B \) and \( C \) are given.

**Problem 5.**

(a)

\[
E[X + Y] = \sum x y P_{XY}(x,y) \sum x y P_{XY}(x,y)
\]

\[
= \sum x P_X(x) \sum y P_Y(y)
\]

\[
= E[X] + E[Y].
\]

Note that independence is not necessary here and that the argument extends to non-discrete variables if the expectation exists.

(b)

\[
E[XY] = \sum x y P_{XY}(x,y)
\]

\[
= \sum x y P_X(x) P_Y(y)
\]

\[
= E[X] E[Y].
\]
Note that the statistical independence was used on the second line. Let \( X \) and \( Y \) take on only the values \( \pm 1 \) and 0. An example of uncorrelated but dependent variables is

\[
P_{XY}(1, 0) = P_{XY}(0, 1) = P_{XY}(-1, 0) = P_{XY}(0, -1) = \frac{1}{4}.
\]

An example of correlated and dependent variables is

\[
P_{XY}(1, 1) = P_{XY}(-1, -1) = \frac{1}{2}.
\]

(c) Using (a), we have

\[
\sigma^2_{X+Y} = E\left[ (X - E[X] + Y - E[Y])^2 \right] = E[(X - E[X])^2] + 2E[(X - E[X])(Y - E[Y])] + E[(Y - E[Y])^2].
\]

The middle term, from (a), is \( 2(E[XY] - E[X]E[Y]) \). For uncorrelated variables that is zero, leaving us with \( \sigma^2_{X+Y} = \sigma_X^2 + \sigma_Y^2 \).

**Problem 6.** We solve the problem for a general vehicle with \( n \) wheels.

(a) Out of \( n! \) possible orderings \((n - 1)!\) has the tyre 1 in its original place. Thus tyre 1 is installed in its original position with probability \( \frac{1}{n} \).

(b) All tyres end up in their original position in only 1 of the \( n! \) orders. Thus the probability of this event is \( \frac{1}{n!} \).

(c) Let \( X_i \) be the indicator random variable that tyre \( i \) is installed in its original position, so that the number of tyres installed in their original positions is \( N = \sum_{i=1}^{n} X_i \). By (a), \( E[X_i] = \frac{1}{n} \). By the linearity of expectation, \( E[N] = n(\frac{1}{n}) = 1 \). Note that the linearity of the expectation holds even if the \( X_i \)’s are not independent (as it is in this case).

(e) Let \( A_i \) be the event that the \( i \)th tyre remains in its original position. Then, the event we are interested in is the complement of the event \( \bigcap_i A_i \) and thus has probability \( 1 - \Pr(\bigcap_i A_i) \). Furthermore, by the inclusion/exclusion formula,

\[
\Pr(\bigcap_i A_i) = \sum_{i} \Pr(A_i) - \sum_{i < j} \Pr(A_i \cap A_j) + \sum_{i < j < k} \Pr(A_i \cap A_j \cap A_k) - \ldots \nabla
\]

The \( j \)th sum above consists of \( \binom{n}{j} \) terms, each term having the value \( \Pr(A_1 \cap \cdots \cap A_j) \). Note that this is the probability of the event that tyres 1 through \( j \) have remained in their original positions, and equals \( (n - j)!/n! \). Consequently,

\[
\Pr(\bigcap_i A_i) = \sum_{j=1}^{n} \binom{n}{j} (-1)^{j-1} \frac{n - j)!}{n!} = \sum_{j=1}^{n} \binom{n}{j} (-1)^{j-1} n^{-j}/j!,
\]

and the event that no tyre remains in its original position has probability

\[
1 - \Pr(\bigcap_i A_i) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j j!.\n\]

(For the case \( n = 4 \), the value is \( 3/8 \).)
Problem 7.

(a) Let $A_i$ denote the event that $X_i \neq X$. The event that $X$ does not appear in the inventory is thus

$$A = A_1 \cap \cdots \cap A_n.$$ 

Note that the events $A_1, \ldots, A_n$ are not independent—because they involve the common random variable $X$. However, they become independent when conditioned on the value of $X$, with $P(A_i|X = x) = 1 - p(x)$. Thus,

$$P(A|X = x) = (1 - p(x))^n.$$ 

Consequently

$$P(A) = \sum_x p(x)(1 - p(x))^n.$$ 

(b) With $p$ the uniform distribution on $n$ items, the above value for $P(A)$ equals $(1 - 1/n)^n$.

(c) For $n$ large, $(1 - 1/n)^n$ approaches $1/e \approx 37\%$. 