ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
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Handout 20 Information Theory and Coding
Homework 8 Nov. 10, 2015

PROBLEM 1. *Channels with memory have higher capacity.* Consider a binary symmetric channel with $Y_i = X_i \oplus Z_i$, where $\oplus$ is mod 2 addition, and $X_i, Y_i \in \{0, 1\}$.

Suppose that $\{Z_i\}$ has constant marginal probabilities $\Pr\{Z_i = 1\} = p = 1 - \Pr\{Z_i = 0\}$, but that $Z_1, Z_2, \ldots, Z_n$ are not necessarily independent. Assume that $(Z_1, \ldots, Z_n)$ is independent of the input $(X_1, \ldots, X_n)$. Let $C = \log 2 - H(p, 1-p)$. Show that

$$\max_{p_{X_1, X_2, \ldots, X_n}} I(X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots, Y_n) \geq nC.$$

PROBLEM 2. Consider two discrete memoryless channels. The input alphabet, output alphabet, transition probabilities and capacity of the k'th channel is given by $\mathcal{X}_k$, $\mathcal{Y}_k$, p_k and C_k respectively ($k = 1, 2$). The channels operate independently. A communication system has access to both channels, that is, the effective channel between the transmitter and receiver has input alphabet $\mathcal{X}_1 \times \mathcal{X}_2$, output alphabet $\mathcal{Y}_1 \times \mathcal{Y}_2$ and transition probabilities $p_1(y_1|x_1)p_2(y_2|x_2)$. Find the capacity of this channel.

PROBLEM 3. Let P_1 and P_2 be two channels of input alphabet $\mathcal{X}_1$ and $\mathcal{X}_2$ and of output alphabet $\mathcal{Y}_1$ and $\mathcal{Y}_2$ respectively. Consider a communication scheme where the transmitter chooses the channel (P_1 or P_2) to be used and where the receiver knows which channel were used. This scheme can be formalized by the channel P of input alphabet $\mathcal{X} = (\mathcal{X}_1 \times \{1\}) \cup (\mathcal{X}_2 \times \{2\})$ and of output alphabet $\mathcal{Y} = (\mathcal{Y}_1 \times \{1\}) \cup (\mathcal{Y}_2 \times \{2\})$, which is defined as follows:

$$P(y, k'|x, k) = \begin{cases} P_k(y|x) & \text{if } k' = k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $X = (X_k, K)$ be a random variable in $\mathcal{X}$ which will be the input distribution to the channel P, and let $Y = (Y_k, K) \in \mathcal{Y}$ be the output distribution. Define X_1 as being the random variable in $\mathcal{X}_1$ obtained by conditioning X_k on $K = 1$. Similarly define X_2, Y_1 and Y_2. Let α be the probability that $K = 1$.

(a) Show that $I(X; Y) = h_2(\alpha) + \alpha I(X_1; Y_1) + (1 - \alpha) I(X_2; Y_2)$.

(b) What is the input distribution X that achieves the capacity of P?

(c) Show that the capacity C of P satisfies $2^C = 2^{C_1} + 2^{C_2}$, where C_1 and C_2 are the capacities of P_1 and P_2 respectively.

PROBLEM 4. Show that a cascade of n identical binary symmetric channels,

$$X_0 \to \boxed{\text{BSC \#1}} \to X_1 \to \cdots \to X_{n-1} \to \boxed{\text{BSC \#n}} \to X_n$$

each with raw error probability p, is equivalent to a single BSC with error probability $\frac{1}{2}\big(1 - (1 - 2p)^n\big)$ and hence that $\lim_{n \to \infty} I(X_0; X_n) = 0$ if $p \neq 0, 1$. Thus, if no processing is allowed at the intermediate terminals, the capacity of the cascade tends to zero.

PROBLEM 5. Consider a memoryless channel with transition probability matrix $P_{Y|X}(y|x)$, with $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. For a distribution Q over $\mathcal{X}$, let $I(Q)$ denote the mutual information between the input and the output of the channel when the input distribution is Q. Show that for any two distributions Q and Q' over $\mathcal{X}$,

(a)

$$I(Q') \leq \sum_{x \in \mathcal{X}} Q'(x) \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x) \log \left(\frac{P_{Y|X}(y|x)}{\sum_{x' \in \mathcal{X}} P_{Y|X}(y|x') Q(x')} \right)$$

(b)

$$C \leq \max_{x} \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x) \log \left(\frac{P_{Y|X}(y|x)}{\sum_{x' \in \mathcal{X}} P_{Y|X}(y|x') Q(x')} \right)$$

where C is the capacity of the channel. Notice that this upper bound to the capacity is independent of the maximizing distribution.