Exercise Set 3 : 10-11 March 2016
Calcul Quantique

Exercise 1 Production of Bell states

a) Check the following identity using Dirac’s notation :

\[ |B_{xy}⟩ = (CNOT)(H \otimes I)|x⟩ \otimes |y⟩. \]

where \( x, y \in \{0, 1\} \) and \( |B_{xy}⟩ \) are the Bell states.

b) Represent the corresponding circuit.

c) Represent the circuit corresponding to the inverse identity :

\[ |x⟩ \otimes |y⟩ = (H \otimes I)(CNOT)|B_{xy}⟩ \]

Exercise 2 Construction of a multi-control-U.

Verify that the multi-control-\( U \) :

\[
\begin{align*}
|c_1⟩ & \rightarrow \cdot \rightarrow |c_1⟩ \\
|c_2⟩ & \rightarrow \cdot \rightarrow |c_2⟩ \\
|c_3⟩ & \rightarrow \cdot \rightarrow |c_3⟩ \\
|t⟩ & \rightarrow U \rightarrow U^{c_1c_2c_3}|t⟩
\end{align*}
\]

can be realized with the Toffoli gate (control-control-NOT) a simple simple control-\( U \).

\[
\begin{align*}
|c_1⟩ & \rightarrow \cdot \rightarrow |c_1⟩ \\
|c_2⟩ & \rightarrow \cdot \rightarrow |c_2⟩ \\
|c_3⟩ & \rightarrow \cdot \rightarrow |c_3⟩ \\
|0⟩ & \rightarrow \odot \rightarrow |0⟩ \\
|0⟩ & \rightarrow \odot \rightarrow |0⟩ \\
|c_4⟩ & \rightarrow U \rightarrow U^{c_1c_2c_3}|c_4⟩
\end{align*}
\]

\{ \text{controlled bits} \}
\{ \text{extra working space} \}
\{ \text{target bit} \}
**Exercise 3** Construction of the Toffoli gate from a control-NOT (Indication : long calculation).

Verify that the control-control-NOT also called Toffoli gate:

\[
|c_1\rangle \quad |c_1\rangle \\
|c_2\rangle \quad |c_2\rangle \\
|t \oplus c_1c_2\rangle
\]

is equivalent to the following circuit made of CNOT, H, T and S

**Exercise 4** Unitary representation of a reversible computation.

Classically a Boolean function \( f \) with inputs \((x_1, \ldots, x_n) \in \{0, 1\}^n\) and output in \(\{0, 1\}\) can be computed reversibly as

\[
\tilde{f}(x_1, \ldots, x_n; y) = (x_1, \ldots, x_n; y \oplus f(x_1, \ldots, x_n))
\]

where \(y \in \{0, 1\}\) is a single storage bit.

This can be implemented in a quantum circuit thanks to the following unitary operation

\[
U_f |x_1, \ldots, x_n; y\rangle = |x_1, \ldots, x_n; y \oplus f(x_1, \ldots, x_n)\rangle
\]

a) What is the Hilbert space relevant for this implementation. Prove that \(U_f\) is indeed a unitary matrix.

b) Generalize this discussion to the case where the output of \(f\) in \(\{0, 1\}^m\) (there are \(m\) output bits).