Exercise 1 1-qubit non-classical gate

Consider a gate $G = \begin{pmatrix} \frac{1}{2} + \frac{i}{2} & \frac{1}{2} - \frac{i}{2} \\ \frac{1}{2} - \frac{i}{2} & \frac{1}{2} + \frac{i}{2} \end{pmatrix}$.

(a) Show that $G^2$ is a real matrix.
(b) For $|v\rangle = |0\rangle, |1\rangle$ and $\alpha |0\rangle + \beta |1\rangle$, compute $G|v\rangle$ and $G^2|v\rangle$.
(c) The gate $G$ is a purely quantum logic gate without any classical equivalent. However, $G^2$ is equivalent to a classical logic gate. What classical logic gate is it?

Exercise 2 Superposition of exponentially many component states

Suppose we have $n$ qubits, all prepared in state $|0\rangle$. We want to obtain an equal superposition of $2^n$ component states, given by

$$\frac{1}{\sqrt{2^n}} \sum_{b_1, b_2, \ldots, b_n \in \{0, 1\}^2} |b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_n\rangle.$$  \hspace{1cm} (1)

For example, when $n = 2$, we want to obtain

$$\frac{1}{2} |0\rangle \otimes |0\rangle + \frac{1}{2} |0\rangle \otimes |1\rangle + \frac{1}{2} |1\rangle \otimes |0\rangle + \frac{1}{2} |1\rangle \otimes |1\rangle.$$  \hspace{1cm}

Can you create a circuit that outputs (1)? Start thinking with $n = 1, 2$.

Exercise 3 SWAP · controlled-\textit{U} · SWAP

Let $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$. The controlled-$U$ gate is given by the following circuit:

\begin{align*}
|c\rangle & \quad \bullet \quad |c\rangle \\
|t\rangle & \quad \text{[U]} \quad U^c|t\rangle = \begin{cases} 
|t\rangle & \text{if } c = 0 \\
U|t\rangle & \text{if } c = 1
\end{cases}
\end{align*}

(a) Show that the controlled-$U$ has the following matrix representation:

$$\text{controlled-}U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{11} & U_{12} \\ 0 & 0 & U_{21} & U_{22} \end{pmatrix}$$
(b) What is the circuit for SWAP · controlled-$U$ · SWAP?

(c) Check that SWAP · controlled-$U$ · SWAP has the following matrix representation:

$$\text{SWAP} \cdot \text{controlled-$U$} \cdot \text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & U_{11} & 0 & U_{12} \\ 0 & 0 & 1 & 0 \\ 0 & U_{21} & 0 & U_{22} \end{pmatrix}$$

**Exercise 4** Controlled-controlled-$U$

In the last exercise, we have seen the construction of multi-controlled-$U$ gate with the Toffoli gates and a simple controlled-$U$ gate. This time, we are going to see a different construction for controlled-controlled-$U$ gate.

Let $V$ be any quantum gate such that $V^2 = U$. Prove the following circuit identity.

![Diagram of controlled-controlled-$U$ gate]

\[
\begin{array}{c}
\quad \equiv \quad \\
U \quad V \quad V^2 \quad V
\end{array}
\]