Problem 1 (Statistics of AMP and IST un-thresholded estimates). Consider a sparse signal $x_0$ with $n$ iid components distributed as $(1 - \epsilon)\delta(x_0) + \frac{\epsilon}{2}\delta(x - 1) + \frac{\epsilon}{2}\delta(x + 1)$. Generate $m$ noisy measurements $y = \frac{1}{\sqrt{m}} \tilde{A} x + z$ where $\tilde{A}_{ni}$ are iid uniform in $\{+1, -1\}$ and $z_a$ are iid Gaussian zero mean and variance $\sigma^2$.

Consider the AMP iterations
\[
\begin{align*}
\hat{x}_i^{(t+1)} &= \eta(\hat{x}_i^{(t)} + \frac{1}{\sqrt{m}} \sum_{b=1}^{m} \tilde{A}_{bi} r_b^{(t)}; \theta^{(t)}), \\
r_a^{(t)} &= y_a - \frac{1}{\sqrt{m}} \sum_{j=1}^{n} \tilde{A}_{aj} \hat{x}_j^{(t)} + r_a^{t-1} \|\hat{x}^{(t)}\|_0/m,
\end{align*}
\]
with the choice $\theta^{(t)} = \alpha \|x^{(t)}\|_2/\sqrt{m}$. In class we derived through heuristic means that the $i$-th component, given $x_0$, of the un-thresholded estimate
\[
\hat{x}_i^{(t)} + \frac{1}{\sqrt{m}} \sum_{b=1}^{m} \tilde{A}_{bi} r_b^{(t)},
\]
has Gaussian statistics. The mean is $x_{0i}$ and the variance $\sigma^2 + \left(\hat{\tau}\right)^{(2)}$ where $\left(\hat{\tau}\right)^{(2)} = \|x^{(t)} - x_0\|_2^2/n$.

Perform an experiment to check this numerically. Compute also the statistics of the un-thresholded estimate for the IST iterations, i.e. when the Onsager term $r_a^{t-1} \|\hat{x}^{(t)}\|_0/m$ is removed. Compare the two histograms.

Indications: Fix a signal realization $x_0$. Try $n = 4000$, $m = 2000$, $\epsilon = 0.125$ and 40 instances for $A$ and $z$. Try various values for $\sigma$ and $\alpha$. Look at the $i$-th components of the un-thresholded estimate for components such that say $x_{0i} = +1$ (or $-1$, or 0).