SOLUTION 1.

(a) With the observation Y being Y_2,

$$f_{Y|X}(y|+1) = \frac{1}{\sqrt{2\pi}} \exp(-(y-1)^2/2) \quad \text{and} \quad f_{Y|X}(y|-1) = \frac{1}{\sqrt{2\pi}} \exp(-(y+1)^2/2)$$

Thus the MAP rule will decide $+1$ or -1 according to $p \exp(-(y-1)^2/2)$ or $(1-p)\exp(-(y+1)^2/2)$ being larger. This can be implemented simply by comparing y to the threshold $\frac{1}{2}\log[(1-p)/p]$ and deciding $+1$ if y is larger, and -1 otherwise.

(b) Observe that

$$\begin{aligned} f_{Y_1Y_2|X}(y_1, y_2|+1) &= \frac{1}{2}\mathbf{1}\{y_1 \in [0,2]\}\frac{1}{\sqrt{2\pi}} \exp(-(y_2-1)^2/2) \\ f_{Y_1Y_2|X}(y_1, y_2|-1) &= \frac{1}{4}\mathbf{1}\{y_1 \in [-3,1]\}\frac{1}{\sqrt{2\pi}} \exp(-(y_2+1)^2/2). \end{aligned}$$

With

$$\begin{aligned} g_{+1}(u, y_2) &= \frac{1}{2}\mathbf{1}\{u \geq 0\}\frac{1}{\sqrt{2\pi}} \exp(-(y_2-1)^2/2) \\ g_{-1}(u, y_2) &= \frac{1}{4}\mathbf{1}\{u \leq 0\}\frac{1}{\sqrt{2\pi}} \exp(-(y_2+1)^2/2) \\ h(y_1, y_2) &= \mathbf{1}\{-3 \leq y_1 \leq 2\} \end{aligned}$$

we find $f_{Y_1Y_2|X}(y_1, y_2|x) = g_x(u, y_2)h(y_1, y_2)$ and the Fisher–Neyman theorem lets us conclude that $t = (u, y_2)$ is a sufficient statistic.

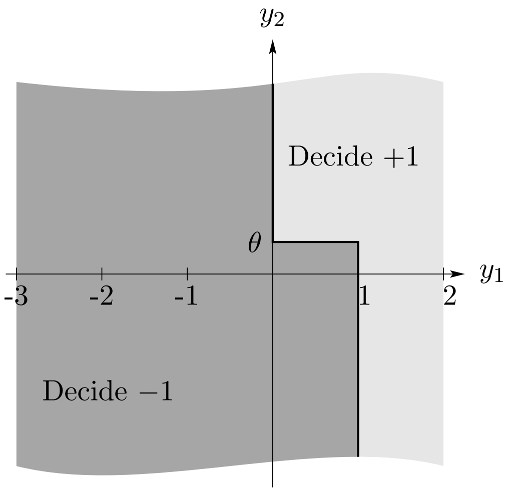
(c) The MAP rule minimizes the error probability and is given by the likelihood ratio test

$$\Lambda(y_1, y_2) = \log \frac{f_{Y_1Y_2|X}(y_1, y_2|+1)}{f_{Y_1Y_2|X}(y_1, y_2|-1)} \underset{-1}{\overset{+1}{\gtrless}} \log \frac{1-p}{p}$$

Note that

$$\Lambda(y_1, y_2) = \begin{cases} +\infty & 1 < y_1 \leq 2 \\ 2y_2 + \log 2 & 0 \leq y_1 \leq 1 \\ -\infty & -3 \leq y_1 < 0 \end{cases}$$

So the decision region looks as follows (with $\theta = \frac{1}{2}\log\frac{1-p}{2p}$):



(d) When -1 is sent an error will happen either when $y_1 > 1$ or when $0 \le y_1 \le 1$ and $y_2 \ge \theta$. The first of these cannot happen, and the second happens with probability $\frac{1}{4}Q(1+\theta)$.

When $+1$ is sent an error will happen either when $y_1 < 0$ or when $0 \le y_1 \le 1$ and $y_2 \le \theta$. The first of these cannot happen, and the second happens with probability $\frac{1}{2}Q(1-\theta)$.

So the error probability is given by

$$\frac{1-p}{4}Q(\theta+1) + \frac{p}{2}Q(1-\theta)$$

with $\theta = \frac{1}{2}\log\frac{1-p}{2p}$.

Solution 2. Note that the decision statistic Y is given by

$$Y = \begin{cases} (w*h)(t_0) + Z & 1 \text{ is sent} \\ Z & 0 \text{ is sent} \end{cases}$$

where $Z = (N*h)(t_0)$ is $\mathcal{N}(0, \|h\|^2)$.

(a) With the given choice of $h(t)$ and t_0 we see find $(w*h)(t_0) = 1/6$ and $\|h\|^2 = 4/3$, so the decision statistic Y is given by

$$Y = \begin{cases} 1/6 + Z & 1 \text{ is sent} \\ Z & 0 \text{ is sent} \end{cases}$$

and the ML rule will compare Y to the threshold $1/12$. The resulting error probability is then $Q(1/(12\|h\|)) = Q(\sqrt{3}/24)$.

(b) Yes. With the choice $t_0 = 4$ we would be implementing the matched filter which we know to be optimal. Indeed the decision statistic will be

$$Y = \begin{cases} 4/3 + Z & 1 \text{ is sent} \\ Z & 0 \text{ is sent} \end{cases}$$

with Z as above, and the error probability will be $Q((2/3)/\|h\|) = Q(1/\sqrt{3})$.

(c) With this new choice of $h(t)$ and t_0 we find $(w * h)(t_0) = 1$ and $\|h\|^2 = 2$. The ML rule will then compare Y to the threshold $1/2$ and the resulting error probability will be $Q(1/(2\|h\|)) = Q(1/\sqrt{8})$.

(d) Similar to the idea in Homework 6 Problem 1, we can sample the filter output at instants $t_0 = 2$ and $t_1 = 4$ and subtract them from each other. The resulting decision statistic will be

$$Y = \begin{cases} 2 + Z & 1 \text{ is sent} \\ Z & 0 \text{ is sent} \end{cases}$$

with Z being $\mathcal{N}(0, 4)$. The resulting error probability will be $Q(1/2)$.

SOLUTION 3.

(a) On this basis the representations of the signals are $c_1 = [2\ 0\ 0\ 2]$, $c_2 = [0\ 2\ 2\ 0]$, $c_3 = [2\ 0\ 2\ 0]$, $c_4 = [0\ 2\ 0\ 2]$.

(b) (2 pts) The union bound is expressed in terms of the pairwise distances d_{ij} between the signals as

$$P(\text{error}|i) \leq \sum_{j \neq i} Q(d_{ij}/2\sigma).$$

From (a) we observe that $d_{12}^2 = d_{34}^2 = 16$, $d_{13}^2 = d_{14}^2 = d_{23}^2 = d_{24}^2 = 8$ as we obtain

$$P(\text{error}|i) \leq 2Q(2/\sqrt{N_0}) + Q(2\sqrt{2}/\sqrt{N_0})$$

Since the right hand side is the same for each i it also bounds the average error probability.

(c) To obtain the minimum energy constellation we need to subtract from each signal $[w_1(t) + w_2(t) + w_3(t) + w_4(t)]/4 = \mathbf{1}\{0 \leq t \leq 4\}$. The resulting signals look exactly as the original ones except for being shifted down by 1 unit.

(d) Note that in the new signal set $\tilde{w}_2(t) = -\tilde{w}_1(t)$ and $\tilde{w}_4(t) = -\tilde{w}_3(t)$. Furthermore the signals $\tilde{w}_1(t)$ and $\tilde{w}_3(t)$ are orthogonal. Thus the new signal space is two dimensional, and the Gram–Schmidt procedure will produce the orthonormal basis $\tilde{\psi}_1(t) = \frac{\tilde{w}_1(t)}{\|\tilde{w}_1\|} = \frac{1}{2}\tilde{w}_1(t)$ and $\tilde{\psi}_2(t) = \frac{\tilde{w}_3(t)}{\|\tilde{w}_3\|} = \frac{1}{2}\tilde{w}_3(t)$.

(e) The new signal set in the new basis is represented by $\tilde{c}_1 = [+2\ 0]$, $\tilde{c}_2 = [-2\ 0]$, $\tilde{c}_3 = [0\ +2]$, $\tilde{c}_4 = [0\ -2]$ and thus is the 4-QAM signal set (rotated by 45 degrees). The error probability of this set is

$$P(\text{error}) = 1 - \left[1 - Q(2/\sqrt{N_0})\right]^2 = 2Q(2/\sqrt{N_0}) - Q(2/\sqrt{N_0})^2.$$

(f) Since translations of a signal set do not change the probability of error, the probability of error of the receiver in (b) is equal to the result we found in (e).