ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 21 Principles of Digital Communications
Problem Set 9 Apr. 29, 2015

PROBLEM 1. *(Properties of the Self-Similarity Function)* Prove the following properties of the self-similarity function. Recall that the self-similarity function of an $\mathcal{L}_2$ pulse $\xi(t)$ is

$$R_\xi(\tau) = \int_{-\infty}^{\infty} \xi(t+\tau)\xi^*(t)\, dt.$$

(a) *Value at zero:*

$$R_\xi(\tau) \le R_\xi(0) = \|\xi\|^2, \qquad \tau \in \mathbb{R}.$$

(b) *Conjugate symmetry:*

$$R_\xi(-\tau) = R_\xi^*(\tau), \qquad \tau \in \mathbb{R}.$$

(c) *Convolution representation:*

$$R_\xi(\tau) = \xi(\tau) \star \xi^*(-\tau), \qquad \tau \in \mathbb{R}.$$

(d) *Fourier relationship:*

$$R_\xi(\tau) \text{ is the inverse Fourier transform of } |\xi(f)|^2.$$

Note: The fact that $\xi_{\mathcal{F}}(f)$ is in $\mathcal{L}_2$ implies that $|\xi_{\mathcal{F}}(f)|^2$ is in $\mathcal{L}_1$. The Fourier inverse of an $\mathcal{L}_1$ function is continuous. Hence $R_\xi(\tau)$ is continuous.

PROBLEM 2. *(Matched Filter Basics)* Let

$$w(t) = \sum_{k=1}^{K} d_k \psi(t - kT)$$

be a transmitted signal where $\psi(t)$ is a real-valued pulse that satisfies

$$\int_{-\infty}^{\infty} \psi(t)\psi(t-kT)dt = \begin{cases} 0, & k \neq 0 \\ 1, & k = 0, \end{cases}$$

and $d_k \in \{-1, 1\}$.

(a) Suppose that $w(t)$ is filtered at the receiver by the matched filter with impulse response $\psi(-t)$. Show that the filter output $y(t)$ sampled at mT, $m \in \mathbb{Z}$, yields $y(mT) = d_m$, for $1 \le m \le K$.

(b) Now suppose that the (noiseless) channel outputs the input plus a delayed and scaled replica of the input. That is, the channel's impulse response takes the form $h(t) = \delta(t) + \rho\delta(t-T)$ for some T and some $\rho \in [-1, 1]$. The transmitted signal $w(t)$ is filtered by $h(t)$, then filtered at the receiver by $\psi(-t)$. The resulting waveform $\tilde{y}(t)$ is again sampled at multiples of T. Determine the samples $\tilde{y}(mT)$, for $1 \le m \le K$.

(c) Suppose that the k-th received sample is $Y_k = d_k + \alpha d_{k-1} + Z_k$, where $Z_k \sim \mathcal{N}(0, \sigma^2)$ and $0 \le \alpha < 1$ is a constant. d_k and d_{k-1} are realizations of independent random variables that take on the values 1 and -1 with equal probability. Suppose that the receiver decides $\hat{d}_k = 1$ if $Y_k > 0$, and decides $\hat{d}_k = -1$ otherwise. Find the probability of error for this receiver.

PROBLEM 3. *(Differential Encoding)* For many years, telephone companies built their networks on *twisted pairs*. This is a twisted pair of copper wires invented by Alexander Graham Bell in 1881 as a means to mitigate the effect of electromagnetic interference. In essence, an alternating magnetic field induces an electric field in a loop. This applies also to the loop created by two parallel wires connected at both ends. If the wire is twisted, the electric field components that build up along the wire alternate polarity and tend to cancel out one another. If we swap the two contacts at one end of the cable, the signal's polarity at one end is the opposite of that on the other end. Differential encoding is a technique for encoding the information in such a way that it makes no difference when the polarity is inverted. The differential encoder takes the data sequence $\{D_i\}_{i=1}^n$, here assumed to have independent and uniformly distributed components taking value in $\{0, 1\}$, and produces the symbol sequence $\{X_i\}_{i=1}^n$ according to the following encoding rule:

$$X_i = \begin{cases} X_{i-1}, & D_i = 0, \\ -X_{i-1}, & D_i = 1, \end{cases}$$

where $X_0 = \sqrt{\mathcal{E}}$ by convention. Suppose that the symbol sequence is used to form

$$X(t) = \sum_{i=1}^{n} X_i \psi(t - iT),$$

where $\psi(t)$ is normalized and orthogonal to its T-spaced time translates. The signal is sent over the AWGN channel of power spectral density $N_0/2$ and at the receiver is passed through the matched filter of impulse response $\psi^*(-t)$. Let Y_i be the filter output at time iT.

(a) Determine $K_X[k]$, $k \in \mathbb{Z}$, assuming an infinite sequence $\{X_i\}_{i=-\infty}^{\infty}$.

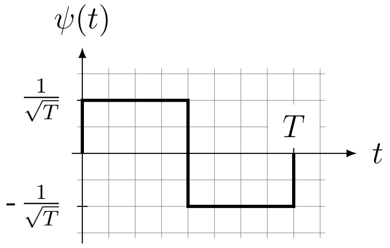
(b) Describe a method to estimate D_i from Y_i and Y_{i-1}, such that the performance is the same if the polarity of Y_i is inverted for all i. We ask for a simple decoder, not necessarily ML.

(c) Determine (or estimate) the error probability of your decoder.

PROBLEM 4. *(Power Spectrum: Manchester Pulse)* Let the random process $X(t)$ be generated as

$$X(t) = \sum_{i=-\infty}^{\infty} X_i \psi(t - iT - \Theta)$$

where $\{X_i\}_{i=-\infty}^{\infty}$ is the input of the waveform former, Θ is uniformly distributed in the interval $[0, T]$, and $\psi(t)$ is the so-called Manchester pulse shown below. The Manchester pulse guarantees that $X(t)$ has at least one transition per symbol, which facilitates the clock recovery at the receiver.



(a) Assume X_i is created from the i.i.d. data sequence $\{D_i\}_{i=-\infty}^{+\infty}$ taking values in $\{\pm 1\}$ with equal probability by setting $X_i = \sqrt{\mathcal{E}} D_i$. Compute the power spectral density of $X(t)$. At which frequencies is the power spectrum of $X(t)$ zero?

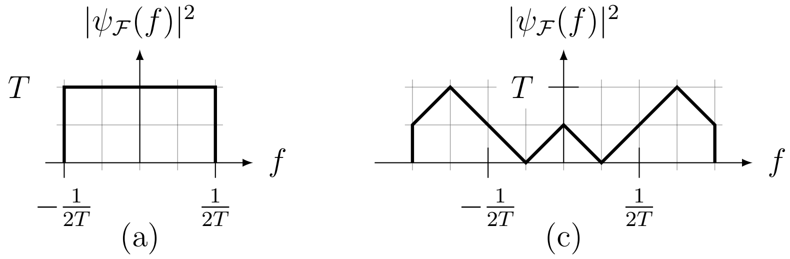
(b) Suppose it is desired to additionally have a zero in the power spectrum of $X(t)$ at $f = \frac{1}{T}$. To this end, one proposes a precoding scheme by letting X_i be

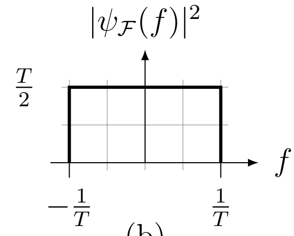
$$X_i = \sqrt{\mathcal{E}}(D_i + \alpha D_{i-1})$$

where α is a *real* constant. Is it possible to choose α to produce the desired frequency null at $f = \frac{1}{T}$? If yes, what are the appropriate values and the resulting power spectrum?

(c) Now assume we want to have zeros at all multiples of $f_0 = \frac{1}{4T}$. Is it possible to have these zeros with an appropriate choice of α in the previous part? If not, then what kind of precoding do you suggest to obtain the desired nulls?

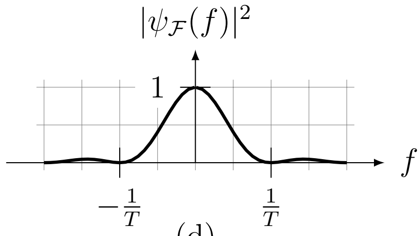
PROBLEM 5. *(Nyquist Criterion)* For each function $|\psi_{\mathcal{F}}(f)|^2$ in the figure below, indicate whether the corresponding pulse $\psi(t)$ has unit norm and/or is orthogonal to its time-translates by multiples of T. The function in (d) is $\text{sinc}^2(fT)$.





$|\psi_{\mathcal{F}}(f)|^2$
$\frac{T}{2}$
f
$-\frac{1}{T}$
$\frac{1}{T}$

(b)



$|\psi_{\mathcal{F}}(f)|^2$
1
f
$-\frac{1}{T}$
$\frac{1}{T}$

(d)