Solution 1.

(a) In this case all components of $Y$ except the first will contain only white Gaussian noise:

\[ Y_1 = \sqrt{E} + Z_1 \]
\[ \forall j = 2, \ldots, m, \ Y_j = Z_j, \quad Z_j \sim \mathcal{N}(0, \sigma^2). \]

(b) This is the event that the receiver declares $\hat{H} = 1$, since only $Y_1$ is larger than the threshold.

(c) \[
P_e = \Pr \{(E_1 \cap E_2^c \cap E_3^c \cap \ldots \cap E_m^c)^c\} = \Pr \{E_1^c \cup E_2 \cup E_3 \cup \ldots \cup E_m\}
\]
\[
\leq Q \left( \frac{(1 - \alpha)\sqrt{E}}{\sigma} \right) + (m - 1)Q \left( \frac{\alpha\sqrt{E}}{\sigma} \right),
\]
where the inequality follows from the union bound.

(d) Taking the hints given in the problem, the above expression can be written as:

\[
P_e \leq \frac{1}{2} \left( e^{-\frac{(1-\alpha)^2E}{2\sigma^2}} + e^{\ln m e^{-\frac{\alpha^2E}{2\sigma^2}}} \right)
\]
\[
= \frac{1}{2} \left( e^{-\frac{(1-\alpha)^2E}{2\sigma^2}} + e^{\ln m(1-\frac{E_b}{2\sigma^2})e^{\alpha^2\log_2 e}} \right).
\]

The first term in the sum goes to zero as $E$ grows, but the second term only diminishes if \[ 1 - \frac{E_b}{2\sigma^2} \alpha^2 \log_2 e < 0, \] i.e., if

\[
\frac{E_b}{\sigma^2} > \frac{2 \ln 2}{\alpha^2}.
\]

Solution 2. First we compute $T_s$, which is the duration of one bit:

\[
T_s = \frac{1}{1 \text{ Mbps}} = 10^{-6} \text{ s}.
\]

Now, we can calculate the energy of the signal (i.e., the energy per bit), which is the same for every $j$:

\[
E_b = b^2 T_s.
\]
The bit error probability is given by $Q \left( \frac{\sqrt{E_b}}{\sigma} \right)$. In our case $\sigma = \sqrt{N_0/2} = 10^{-1}$, thus we need to solve

$$10^{-5} = Q \left( \frac{b10^{-3}}{10^{-1}} \right) = Q \left( b10^{-2} \right),$$

hence $b = Q^{-1}(10^{-5}) \times 10^2 \approx 426.5$.

**Solution 3.**

(a) There are various possibilities to choose an orthogonal basis. One is $\phi_1(t) = \frac{w_0(t)}{||w_0||} = \sqrt{\frac{1}{T_s}} w_0(t)$ and $\phi_2(t) = \frac{w_2(t)}{||w_2||} = \sqrt{\frac{1}{T_s}} w_2(t)$. Another choice, that we prefer and will be our choice in this solution is

$$\psi_1(t) = \sqrt{\frac{2}{T_s}} 1_{[0, T_s]}(t)$$
$$\psi_2(t) = \sqrt{\frac{2}{T_s}} 1_{[T_s, 2T_s]}(t).$$

With the latter choice the signal space (shown in the figure below) is

$$w_0 = \sqrt{\frac{T_s}{2}} (1, 1)^T$$
$$w_2 = \sqrt{\frac{T_s}{2}} (1, -1)^T$$
$$w_1 = \sqrt{\frac{T_s}{2}} (-1, -1)^T$$
$$w_3 = \sqrt{\frac{T_s}{2}} (-1, 1)^T$$

(b) $U_0 \in \{ \pm 1 \}$ and $U_1 \in \{ \pm 1 \}$ are mapped into

$$U_0 \sqrt{\frac{T_s}{2}} \psi_1(t) + U_1 \sqrt{\frac{T_s}{2}} \psi_2(t).$$

The mapping is shown here:
The mapping is such that neighboring points differ by one bit. This minimizes the bit-
error probability since when we make an error chances are that we choose a neighbor
of the correct symbol. Notice that we may decode each bit independently. In fact
the first bit is decoded to a 1 iff the observation is to the right of the vertical axis
and the second bit is 1 iff it is above the horizontal axis. The bit error probability is
therefore
\[ P_b = Q\left(\frac{\sqrt{T_s/2}}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{T_s}{N_0}}\right). \]

(c) Notice that \( \psi_2(t) = \psi_1(t - T_s/2) \). Hence one matched filter is enough. The receiver
block diagram is as follows:

\[ R(t) \xrightarrow{\psi_1(T_s/2 - t)} t = T_s/2 \quad Y_1 \xrightarrow{t = T_s} \hat{U}_1 \]
\[ t = T_s \quad Y_2 \xrightarrow{\text{threshold at 0}} \hat{U}_2 \]

(d) \( E_b = \frac{E_s}{2} = \frac{T_s}{2} \) and the power is \( \frac{E_s}{T_s} = 1 \).

**Solution 4.**

(a) The average energy is
\[ \int_{-\infty}^{\infty} |w_i(t)|^2 \, dt = \frac{2E}{T} \int_0^T \cos^2(2\pi(f_c + i\Delta f)t) \, dt \]
\[ = \frac{2E}{T} \left[ \frac{t}{2} + \frac{\sin(2\pi(f_c + i\Delta f)t) \cos(2\pi(f_c + i\Delta f)t)}{4\pi(f_c + i\Delta f)} \right]_0^T = E. \]

(b) Orthogonality requires
\[ E_s^2 \int_0^T \cos(2\pi(f_c + i\Delta f)t) \cos(2\pi(f_c + j\Delta f)t) \, dt = 0, \]
for every $i \neq j$. Using the trigonometric identity $\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha + \beta) + \frac{1}{2}\cos(\alpha - \beta)$, an equivalent condition is

$$\frac{E}{T} \int_0^T \left[ \cos(2\pi(i-j)\Delta f t) + \cos(2\pi(2f_c + (i+j)\Delta f) t) \right] dt = 0.$$  

Integrating we obtain

$$\frac{E}{T} \left[ \frac{\sin(2\pi(i-j)\Delta f T)}{2\pi(i-j)\Delta f} + \frac{\sin(2\pi(2f_c + (i+j)\Delta f) T)}{2\pi(2f_c + (i+j)\Delta f)} \right] = 0.$$  

As $f_c T$ is assumed to be an integer, the result can be simplified to

$$\frac{E}{T} \left[ \frac{\sin(2\pi(i-j)\Delta f T)}{2\pi(i-j)\Delta f} + \frac{\sin(2\pi(2f_c + (i+j)\Delta f) T)}{2\pi(2f_c + (i+j)\Delta f)} \right] = 0.$$  

As $i$ and $j$ are integer, this is satisfied for $i \neq j$ if and only if $2\pi\Delta f T$ is an integer multiple of $\pi$. Hence, we obtain the minimum value of $\Delta f$ if $2\pi\Delta f T = \pi$ which gives $\Delta f = \frac{1}{2}f_c$.

(c) Proceeding similarly, we will have orthogonality if and only if

$$\frac{E}{T} \left[ \frac{\sin(2\pi(i-j)\Delta f T + \theta_i - \theta_j)}{2\pi(i-j)\Delta f} + \frac{\sin(2\pi(2f_c + (i+j)\Delta f) T + \theta_i + \theta_j)}{2\pi(2f_c + (i+j)\Delta f)} \right] = 0.$$  

In this case we see that both parts become zero if and only if $2\pi\Delta f T$ is an even multiple of $\pi$, meaning that the smallest $\Delta f$ is $\Delta f = \frac{1}{T}$ which is twice the minimum frequency separation needed in the previous part. Hence, the cost of phase uncertainty is a bandwidth expansion by a factor of 2.

(d) The condition for essential orthogonality is that

$$\frac{E}{T} \left[ \frac{\sin(2\pi(i-j)\Delta f T + \theta_i - \theta_j)}{2\pi(i-j)\Delta f} + \frac{\sin(2\pi(2f_c + (i+j)\Delta f) T + \theta_i + \theta_j)}{2\pi(2f_c + (i+j)\Delta f)} \right] = 0.$$  

is small compared to the signal’s energy $E$. The first term vanishes if $\Delta f = \frac{1}{T}$. The second term is very small compared to $E$ if $f_c T \gg 1$.

(e) We have $m$ signals separated by $\Delta f$. The approximate bandwidth is $m\Delta f$. This means bandwidth $\frac{2^k}{2T}$ without random phase, and bandwidth $\frac{2^k}{T}$ with random phase. We see that in both cases, $WT$ is proportional to $2^k$, i.e. it grows exponentially with $k$.  

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SOLUTION 5.

(a) The block diagram is shown below.

(b) Given $A = a$, the distance of signals is $2a\sqrt{E_b}$, hence

$$P_e(a) = Q\left(a\sqrt{\frac{2E_b}{N_0}}\right).$$

(c)

$$P_f = \mathbb{E}[P_e(A)] = \int_0^\infty Q\left(a\sqrt{\frac{2E_b}{N_0}}\right) 2ae^{-a^2} \, da.$$  

We integrate by parts, noting that $\int 2ae^{-a^2} \, da = -e^{-a^2}$:

$$P_f = -Q\left(a\sqrt{\frac{2E_b}{N_0}}\right) e^{-a^2}\bigg|_0^\infty + \int_0^\infty Q'\left(a\sqrt{\frac{2E_b}{N_0}}\right) e^{-a^2} \, da.$$  

Taking the derivative of an integral with respect to the lower boundary gives the negative of the value of the integrand evaluated at the lower boundary, i.e.

$$Q'(x) = -\frac{1}{\sqrt{2\pi}} e^{-x^2}.$$  

Thus, for the derivative of $Q\left(a\sqrt{\frac{2E_b}{N_0}}\right)$ with respect to $a$, we can write

$$\frac{d}{da} Q\left(a\sqrt{\frac{2E_b}{N_0}}\right) = -\frac{1}{\sqrt{2\pi}} e^{-a^2} \frac{2E_b}{N_0} \sqrt{\frac{2E_b}{N_0}}.$$  

Plugging this in, we find

$$P_f = \frac{1}{2} - \int_0^\infty \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2E_b}{N_0}} e^{-a^2} \left(\frac{E_b}{N_0} + 1\right) \, da,$$

which we now reshape to make it an integral over a Gaussian density, as follows:

$$P_f = \frac{1}{2} - \int_0^\infty \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2E_b}{N_0}} e^{-a^2} \left(\frac{E_b}{N_0} + 1\right) \, da.$$
Now, it is clear that the integral evaluates to one half (since the integral is only over half of the real line), and we find
\[ P_f = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\mathcal{E}_b/N_0}{1 + \mathcal{E}_b/N_0}} = \frac{1}{2} \left( 1 - \sqrt{\frac{\mathcal{E}_b/N_0}{1 + \mathcal{E}_b/N_0}} \right). \]

(d) Let \( \sigma = \frac{1}{\sqrt{2}} \), then
\[ m = \mathbb{E}[A] = \int_0^\infty 2a^2 e^{-a^2} \, da = 2\sqrt{\pi} \int_0^\infty a^2 \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{a^2}{2\sigma^2}} \, da = \sqrt{\pi} \sigma^2 = \frac{\sqrt{\pi}}{2}. \]

Thus, using the formula from part (b):
\[ P_e(m) = Q \left( \sqrt{\frac{2\mathcal{E}_b}{N_0}} \frac{m}{\sqrt{\pi} \sigma^2} \right) = Q \left( \sqrt{\frac{\mathcal{E}_b}{N_0}} \right). \]

For the given example we get
\[ \frac{\mathcal{E}_b}{N_0} = \frac{2 (Q^{-1}(10^{-5}))^2}{\pi} \approx 10.6 \text{ dB}. \]

For the fading we use the result of part (c) to get
\[ \frac{\mathcal{E}_b}{N_0} = \frac{(1 - 2 \cdot 10^{-5})^2}{1 - (1 - 2 \cdot 10^{-5})^2} \approx 44 \text{ dB}. \]

The difference is quite significant! It is clear that this behavior is fundamentally different from the non-fading case.

**Solution 6.**

(a) We pass \( R(t) \) through a whitening filter \( h(t) \) such that the output \( R'(t) \) looks like the output of an AWGN channel. After this step we are facing a familiar situation and can implement a matched filter receiver. The receiver architecture is shown below:
Let \( N'(t) = \int N(\alpha)h(t - \alpha) \, d\alpha \) be the noise at the output of the whitening filter. We want to select the filter \( h(t) \) such that \( \frac{N_0}{2} = G(f)|h(f)|^2 \), i.e.,

\[
|h(f)|^2 = \frac{N_0}{2G(f)}.
\]

The output of the filter is

\[
R'(t) = \int R(\alpha)h(t - \alpha) \, d\alpha = \int w_i(\alpha)h(t - \alpha) \, d\alpha + \int N(\alpha)h(t - \alpha) \, d\alpha = w'_i(t) + N'(t),
\]

where \( N'(t) \) is white Gaussian noise and \( w'_i(t) = \int w_i(\alpha)h(t - \alpha) \, d\alpha \). We need to design the matched filter for the signals \( w'_i(t) \).

(b) To minimize both the noise and the energy of the signal, we need to select an antipodal signal pair that is frequency-limited to \([a, b]\) and has energy \( E \).