ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 15 — Principles of Digital Communications
Solutions to Problem Set 7 — Apr. 7, 2015

Solution 1.

(a) Notice that

$$\|w_0(t)\|^2 = \|w_1(t)\|^2 = \int_0^{2T} w_0^2(t)\ dt = 2T.$$

We apply first the Gram-Schmidt algorithm. We get the first basis vector from the first signal:

$$\psi_0(t) = \frac{w_0(t)}{\|w_0(t)\|} = \begin{cases} \frac{1}{\sqrt{2T}} & t \in [0, 2T] \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\psi_0(t)$ and $w_1(t)$ are orthogonal. Thus the we obtain the second basis vector by normalizing $w_1(t)$:

$$\psi_1(t) = \frac{w_1(t)}{\|w_1(t)\|} = \begin{cases} \frac{1}{\sqrt{2T}} & t \in [0, T] \\ -\frac{1}{\sqrt{2T}} & t \in [T, 2T] \\ 0 & \text{otherwise.} \end{cases}$$

In the $\{\psi_0(t), \psi_1(t)\}$ basis, it is straightforward to see that $c_0 = (\sqrt{2T}, 0)^\mathsf{T}$ and $c_1 = (0, \sqrt{2T})^\mathsf{T}$.

The other basis is the following:

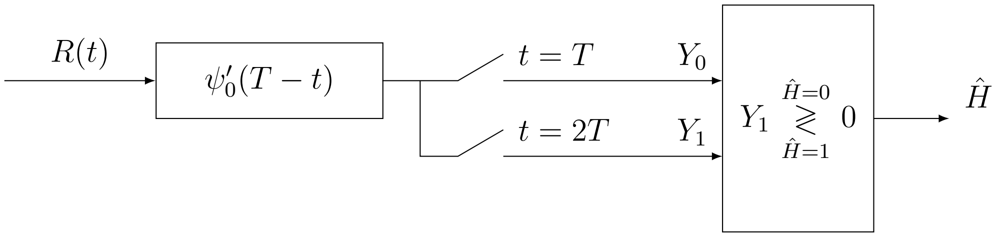
$$\psi_0'(t) = \begin{cases} \frac{1}{\sqrt{T}} & t \in [0, T] \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_1'(t) = \begin{cases} \frac{1}{\sqrt{T}} & t \in [T, 2T] \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\psi_1'(t) = \psi_0'(t - T)$. Hence, one matched filter at the receiver sampled twice suffices to project the received signal onto $\psi_0'(t)$ and $\psi_1'(t)$.

In the $\{\psi_0'(t), \psi_1'(t)\}$ basis, the codewords are $c_0 = (\sqrt{T}, \sqrt{T})^\mathsf{T}$ and $c_1 = (\sqrt{T}, -\sqrt{T})^\mathsf{T}$.

(b) Here is the block diagram of the ML receiver:



Notice that Y_0 is not used. This is not surprising when we look at the signals: For $t \in [0, T]$, the two signals are identical.

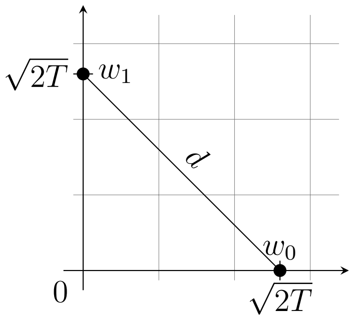
(c) We calculate

$$\|w_0(t) - w_1(t)\| = 2\sqrt{T},$$

hence

$$P_e = Q\left(\frac{\sqrt{T}}{\sqrt{N_0/2}}\right).$$

One can also use the vector representation of the two signals shown below to calculate the probability of error.



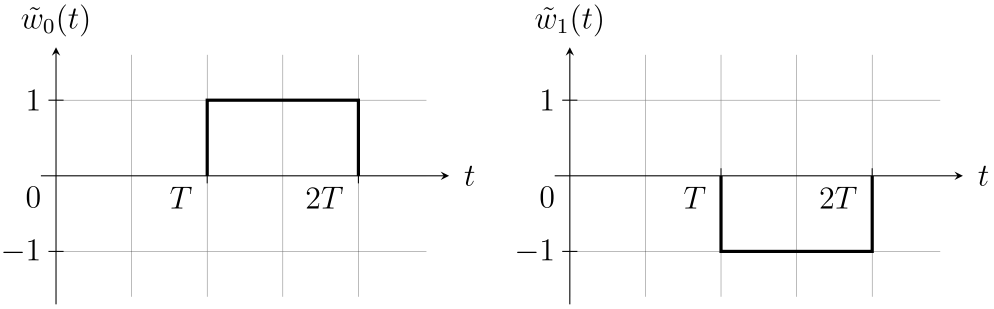
$$P_e = Q\left(\frac{d}{2\sigma}\right) = Q\left(\frac{\sqrt{T}}{\sqrt{N_0/2}}\right).$$

(d) Translating the signal points by any vector will not influence the error probability. However, if the translation vector is the center of mass of the original signal constellation, then the resulting signals will have minimum energy. We compute $v(t) = \frac{1}{2}w_0(t) + \frac{1}{2}w_1(t)$, thus

$$\tilde{w}_0(t) = w_0(t) - v(t) = \begin{cases} 1, & \text{for } t \in [T, 2T] \\ 0, & \text{otherwise} \end{cases}$$

$$\tilde{w}_1(t) = w_1(t) - v(t) = \begin{cases} -1, & \text{for } t \in [T, 2T] \\ 0, & \text{otherwise.} \end{cases}$$

The resulting signal waveforms are shown here:



(e) The new signal constellation is antipodal. One can see that

$$\tilde{w}_0(t) = w_0(t) - v(t) = \frac{1}{2}w_0(t) - \frac{1}{2}w_1(t)$$
$$\tilde{w}_1(t) = w_1(t) - v(t) = \frac{1}{2}w_1(t) - \frac{1}{2}w_0(t) = -\tilde{w}_0(t).$$

This shows that we obtain an antipodal signal constellation regardless of the initial waveforms.

Solution 2.

(a) To find the minimum-energy signal set, we first compute the centroid of the signal set:

$$a = \sum_{j=0}^{m-1} P_H(j)w_j(t) = \frac{1}{m}\sum_{j=0}^{m-1} w_j(t).$$

So

$$\tilde{w}_j(t) = w_j(t) - a = w_j(t) - \frac{1}{m}\sum_{i=0}^{m-1} w_i(t)$$
$$= \frac{m-1}{m}w_j(t) - \frac{1}{m}\sum_{i \neq j} w_i(t).$$

(b) If we write a codebook taking as basis the vectors in $\mathcal{W}$ after translation, then the codeword corresponding to $\tilde{w}_j$ is $\sqrt{\mathcal{E}}\frac{m-1}{m}$ at position j and $-\frac{\sqrt{\mathcal{E}}}{m}$ at all other positions. Clearly $\|\tilde{w}_j(t)\|^2 = (m-1)\frac{\mathcal{E}}{m^2} + \frac{\mathcal{E}}{m^2}(m-1)^2 = \mathcal{E}(1-\frac{1}{m})$. This is independent of j so the average energy is also $\mathcal{E}(1-\frac{1}{m})$. Thus, the energy saving is

$$\mathcal{E} - \tilde{\mathcal{E}} = \frac{1}{m}\mathcal{E}.$$

Alternatively, we could use that $\mathcal{E} - \tilde{\mathcal{E}} = \|a\|^2 = \frac{1}{m}\mathcal{E}$.

(c) Notice that $\sum_{j=0}^{m-1} \tilde{w}_j(t) = 0$ by the definition of $\tilde{w}_j(t), \quad j = 0, 1, \ldots, m-1$. Hence, the m signals $\{\tilde{w}_0(t), \ldots, \tilde{w}_{m-1}(t)\}$ are linearly dependent. This means that their space has dimensionality less than m. We show that any collection of $m-1$ or less is linearly independent. That would prove that the dimensionality of the space $\{\tilde{w}_0(t), \ldots, \tilde{w}_{m-1}(t)\}$ is $m-1$. Without loss of essential generality we consider $\tilde{w}_0(t), \ldots, \tilde{w}_{m-2}(t)$. Assume that $\sum_{j=0}^{m-2} \alpha_j \tilde{w}_j(t) = 0$. Using the definition of $\tilde{w}_j(t)$ we may write

$$\sum_{j=0}^{m-2} \alpha_j \left(w_j(t) - \frac{1}{m} \sum_{i=0}^{m-1} w_i(t) \right) = 0,$$

$$\left(\sum_{j=0}^{m-2} \alpha_j w_j(t) \right) - \left(\frac{1}{m} \sum_{j=0}^{m-2} \alpha_j \right) \sum_{i=0}^{m-1} w_i(t) = 0,$$

$$\left(\sum_{j=0}^{m-2} \alpha_j w_j(t) \right) - \left(\beta \sum_{i=0}^{m-1} w_i(t) \right) = 0,$$

where $\beta = \frac{1}{m} \sum_{j=0}^{m-2} \alpha_j$. Therefore,

$$\sum_{j=0}^{m-2} (\alpha_j - \beta) w_j(t) - \beta w_{m-1}(t) = 0.$$

But $w_0(t), w_1(t), \ldots, w_{m-1}(t)$ is an orthogonal set and this implies $\beta = 0$ and $\alpha_j = \beta = 0,\ j = 0, 1, \ldots, m-2$. Hence $\tilde{w}_j(t),\ j = 0, 1, \ldots, m-2$ are linearly independent. We have proved that the new set spans a space of dimension $m-1$.

Solution 3.

(a) Clearly,

$$\mathcal{E}_s^C(k) = 2^{2k} - 1.$$

(b)

$$a = Q^{-1}\left(\frac{10^{-5}}{2} \right) \approx 4.42.$$

(If we use the approximation $Q(x) \approx \frac{1}{2} e^{-\frac{x^2}{2}}$, we get $a \approx 4.80$.)

(c) We have

$$\mathcal{E}_s^P(k) = \frac{a^2(m^2-1)}{3} = a^2 \frac{2^{2k}-1}{3} \approx 6.5(2^{2k}-1).$$

For comparison, see the following table.

k	$\mathcal{E}_s^P(k)$	$\mathcal{E}_s^C(k)$
1	19.51	3
2	97.58	15
4	1658	255

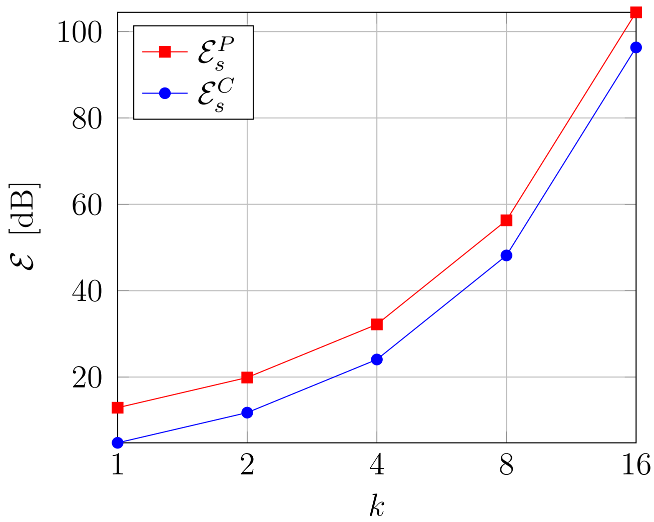
(d) We see that

$$\frac{\mathcal{E}_s^C(k+1)}{\mathcal{E}_s^C(k)} = \frac{\mathcal{E}_s^P(k+1)}{\mathcal{E}_s^P(k)} = \frac{2^{2(k+1)}-1}{2^{2k}-1},$$

thus

$$\lim_{k\to\infty} \frac{\mathcal{E}_s^C(k+1)}{\mathcal{E}_s^C(k)} = \lim_{k\to\infty} \frac{\mathcal{E}_s^P(k+1)}{\mathcal{E}_s^P(k)} = 4.$$

(e) If we send one bit per symbol, then coding allows us to significantly reduce the required energy per symbol. For every additional bit per symbol we need to multiply $\mathcal{E}_s$ by roughly 4 (exactly 4 asymptotically) with or without coding. So as the number of bits per symbol increases, there is essentially a constant gap (in dB) between the energy per symbol required by (uncoded) PAM and that required by the best possible code.



Notice that to keep the error probability at a constant level, we need to increase $\mathcal{E}_s/\sigma^2$ exponentially with the number k of bits per symbol. In Example 4.3 we increase it linearly with k (hence the error probability goes to 1).