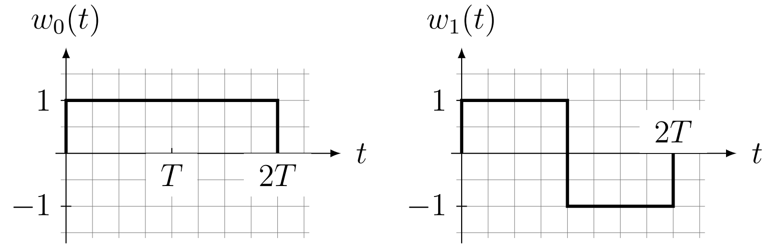
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 14 | Principles of Digital Communications
Problem Set 7 | Apr. 1, 2015

PROBLEM 1. *(Signal Translation)* Consider the signals $w_0(t)$ and $w_1(t)$ shown below, used to communicate one bit across an AWGN channel of power spectral density $N_0/2$.



(a) Determine an orthonormal basis $\{\psi_0(t), \psi_1(t)\}$ for the space spanned by $\{w_0(t), w_1(t)\}$ and find the corresponding codewords c_0 and c_1. Work out two solutions, one obtained via Gram-Schmidt and one in which the second element of the orthonormal basis is a delayed version of the first. Which of the two solutions would you choose if you had to implement the system? (Justify your answer.)

(b) Let X be a uniformly distributed binary random variable that takes values in $\{0, 1\}$. We want to communicate the value of X over an additive white Gaussian noise channel. When $X = 0$, we send $w_0(t)$, and when $X = 1$, we send $w_1(t)$. Draw the block diagram of a ML receiver based on a single matched filter.

(c) Determine the error probability P_e of your receiver as a function of T and N_0.

(d) Find a suitable waveform $v(t)$, such that the new signals $\tilde{w}_0(t) = w_0(t) - v(t)$ and $\tilde{w}_1(t) = w_1(t) - v(t)$ have minimal energy and plot the resulting waveforms.

(e) What is the name of the type of signaling scheme that uses $\tilde{w}_0(t)$ and $\tilde{w}_1(t)$? Argue that one obtains this kind of signaling scheme independently of the initial choice of $w_0(t)$ and $w_1(t)$.

PROBLEM 2. *(Orthogonal Signal Sets)* Consider a set $\mathcal{W} = \{w_0(t), \dots, w_{m-1}(t)\}$ of mutually orthogonal signals with squared norm $\mathcal{E}$ each used with equal probability.

(a) Find the minimum-energy signal set $\tilde{\mathcal{W}} = \{\tilde{w}_0(t), \dots, \tilde{w}_{m-1}(t)\}$ obtained by translating the original set.

(b) Let $\tilde{\mathcal{E}}$ be the average energy of a signal picked at random within $\tilde{\mathcal{W}}$. Determine $\tilde{\mathcal{E}}$ and the energy saving $\mathcal{E} - \tilde{\mathcal{E}}$.

(c) Determine the dimension of the inner-product space spanned by $\tilde{\mathcal{W}}$.

PROBLEM 3. *(Energy Efficiency of Single-Shot PAM)* This exercise, complements what we have leaned in Example 4.3. Consider using a m-PAM constellation:

$$\{\pm a, \pm 3a, \pm 5a, \ldots, \pm(m-1)a\}$$

to communicate across the discrete-time AWGN channel of noise variance $\sigma^2 = 1$. Our goal is to communicate at some level of reliability, say with error probability $P_e \leq 10^{-5}$. We are interested in comparing the energy needed by PAM versus the energy need by a system that operates at *channel capacity*, namely at

$$C = \frac{1}{2}\log_2\left(1 + \frac{\mathcal{E}_s}{\sigma^2}\right)$$

bits per channel use.

(a) Using the capacity formula, determine the energy per symbol, $\mathcal{E}_s^C(k)$ needed to transmit k bits per channel use. At any rate below capacity it is possible to make the error probability arbitrarily small by increasing the codeword length. This implies that there is a way to achieve the desired error probability at energy per symbol $\mathcal{E}_s^C(k)$.

(b) Using single-shot m-PAM, we can achieve an arbitrary small error probability by making the parameter a sufficiently large. As the size m of the constellation increases, the edge effects become negligible and the average error probability approaches $2Q(\frac{a}{\sigma})$ which is the probability of error conditioned on an interior point being transmitted. Find the numerical value of a for which $2Q(\frac{a}{\sigma}) = 10^{-5}$.

(You may use the upper-bound $Q(x) \leq \frac{1}{2}e^{-\frac{x^2}{2}}$ which is tight for large values of x as an approximation to $Q(x)$).

(c) Having determined the value of a, determine the average energy $\mathcal{E}_s^P(k)$ needed by PAM to send k bits at the desired error probability. Compare the numerical values of $\mathcal{E}_s^P(k)$ and $\mathcal{E}_s^C(k)$ for $k = 1, 2, 4$.

Hint: See Equation (4.1) in the book.

(d) Find $\lim_{k\to\infty} \frac{\mathcal{E}_s^C(k+1)}{\mathcal{E}_s^C(k)}$ and $\lim_{k\to\infty} \frac{\mathcal{E}_s^P(k+1)}{\mathcal{E}_s^P(k)}$.

(e) Comment on PAM's efficiency in terms of energy per bit for small and large values of k. Comment also on the relationship between this exercise and Example 4.3.