Problem 1. *(Signal Translation)* Consider the signals \( w_0(t) \) and \( w_1(t) \) shown below, used to communicate one bit across an AWGN channel of power spectral density \( N_0/2 \).

\[ w_0(t) \]
\[ \begin{array}{c}
\cdot \\
T \\
\cdot \\
2T \\
\end{array} \]

\[ w_1(t) \]
\[ \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
2T \\
\end{array} \]

(a) Determine an orthonormal basis \( \{ \psi_0(t), \psi_1(t) \} \) for the space spanned by \( \{ w_0(t), w_1(t) \} \) and find the corresponding codewords \( c_0 \) and \( c_1 \). Work out two solutions, one obtained via Gram-Schmidt and one in which the second element of the orthonormal basis is a delayed version of the first. Which of the two solutions would you choose if you had to implement the system? (Justify your answer.)

(b) Let \( X \) be a uniformly distributed binary random variable that takes values in \{0, 1\}. We want to communicate the value of \( X \) over an additive white Gaussian noise channel. When \( X = 0 \), we send \( w_0(t) \), and when \( X = 1 \), we send \( w_1(t) \). Draw the block diagram of a ML receiver based on a single matched filter.

(c) Determine the error probability \( P_e \) of your receiver as a function of \( T \) and \( N_0 \).

(d) Find a suitable waveform \( v(t) \), such that the new signals \( \tilde{w}_0(t) = w_0(t) - v(t) \) and \( \tilde{w}_1(t) = w_1(t) - v(t) \) have minimal energy and plot the resulting waveforms.

(e) What is the name of the type of signaling scheme that uses \( \tilde{w}_0(t) \) and \( \tilde{w}_1(t) \)? Argue that one obtains this kind of signaling scheme independently of the initial choice of \( w_0(t) \) and \( w_1(t) \).

Problem 2. *(Orthogonal Signal Sets)* Consider a set \( \mathcal{W} = \{ w_0(t), \ldots, w_{m-1}(t) \} \) of mutually orthogonal signals with squared norm \( E \) each used with equal probability.

(a) Find the minimum-energy signal set \( \tilde{\mathcal{W}} = \{ \tilde{w}_0(t), \ldots, \tilde{w}_{m-1}(t) \} \) obtained by translating the original set.

(b) Let \( \tilde{E} \) be the average energy of a signal picked at random within \( \tilde{\mathcal{W}} \). Determine \( \tilde{E} \) and the energy saving \( E - \tilde{E} \).
(c) Determine the dimension of the inner-product space spanned by \( \mathcal{W} \).

**Problem 3. (Energy Efficiency of Single-Shot PAM)** This exercise, complements what we have leaned in Example 4.3. Consider using a \( m \)-PAM constellation:

\[
\{ \pm a, \pm 3a, \pm 5a, \ldots, \pm (m-1)a \}
\]

to communicate across the discrete-time AWGN channel of noise variance \( \sigma^2 = 1 \). Our goal is to communicate at some level of reliability, say with error probability \( P_e \leq 10^{-5} \). We are interested in comparing the energy needed by PAM versus the energy need by a system that operates at *channel capacity*, namely at

\[
C = \frac{1}{2} \log_2 \left( 1 + \frac{E_s}{\sigma^2} \right)
\]

bits per channel use.

(a) Using the capacity formula, determine the energy per symbol, \( E^C_s(k) \) needed to transmit \( k \) bits per channel use. At any rate below capacity it is possible to make the error probability arbitrarily small by increasing the codeword length. This implies that there is a way to achieve the desired error probability at energy per symbol \( E^C_s(k) \).

(b) Using single-shot \( m \)-PAM, we can achieve an arbitrary small error probability by making the parameter \( a \) sufficiently large. As the size \( m \) of the constellation increases, the edge effects become negligible and the average error probability approaches \( 2Q \left( \frac{a}{\sigma} \right) \) which is the probability of error conditioned on an interior point being transmitted. Find the numerical value of \( a \) for which \( 2Q \left( \frac{a}{\sigma} \right) = 10^{-5} \).

(You may use the upper-bound \( Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}} \) which is tight for large values of \( x \) as an approximation to \( Q(x) \)).

(c) Having determined the value of \( a \), determine the average energy \( E^P_s(k) \) needed by PAM to send \( k \) bits at the desired error probability. Compare the numerical values of \( E^P_s(k) \) and \( E^C_s(k) \) for \( k = 1, 2, 4 \).

*Hint: See Equation (4.1) in the book.*

(d) Find \( \lim_{k \to \infty} \frac{E^C_s(k+1)}{E^C_s(k)} \) and \( \lim_{k \to \infty} \frac{E^P_s(k+1)}{E^P_s(k)} \).

(e) Comment on PAM’s efficiency in terms of energy per bit for small and large values of \( k \). Comment also on the relationship between this exercise and Example 4.3.