SOLUTION 1.

(a) The signals that are being sent are \( w_0(t) \) and \( w_1(t) \), where
\[
\begin{align*}
w_0(t) &= w(t), \\
w_1(t) &= 0.
\end{align*}
\]
Clearly the span of \( W = \{w_0, w_1\} \) is one dimensional, and it is spanned by \( w \). Therefore, we can take \( \{\psi\} \) as an orthonormal basis for \( \text{span}(W) \), where \( \psi(t) = \frac{w(t)}{|w|} \).

The codewords corresponding to \( w_0 \) and \( w_1 \) are:
- \( c_0 = \frac{|w|}{\mathbb{R}} \in \mathbb{R} \).
- \( c_1 = 0 \in \mathbb{R} \).

The maximum likelihood receiver for the observable \( R(t) \) uses the matched filter with impulse response \( \phi(t) = \psi(4T - t) = \frac{w(4T - t)}{|w|} \). The \( n \)-tuple former computes \( Y_w = \langle R(t), \psi(t) \rangle \) using the matched filter \( \phi(t) \) as shown below:

\[
\begin{array}{c}
R(t) \quad \phi(t) \quad \frac{t}{4T} \quad Y_w
\end{array}
\]

The maximum likelihood receiver decides \( \hat{H} = 0 \) if \( Y_w > \frac{|w|}{2} \) and \( \hat{H} = 1 \) otherwise.

(b) The probability of error can be directly inferred to be
\[
P_e = Q\left( \frac{|w|}{2\sigma} \right) = Q\left( \frac{|w|}{2\sqrt{\frac{N_0}{2}}} \right) = Q\left( \frac{|w|}{\sqrt{2N_0}} \right).
\]

(c) The question is how to compute \( \langle R(t), \frac{w(t)}{|w|} \rangle \) using \( h(t) \) instead of \( w(t) \). Notice that we have \( w(t) = h(t) - h(t - 2T) \). Therefore,
\[
\langle R(t), \frac{w(t)}{|w|} \rangle = \langle R(t), \frac{h(t)}{|w|} \rangle - \langle R(t), \frac{h(t - 2T)}{|w|} \rangle.
\]

The first term can be obtained via a filter of impulse response \( \frac{h(2T - t)}{|w|} \) and output sampled at \( t = 2T \). The second term can be obtained via a filter of impulse response \( \frac{h((4T - t) - 2T)}{|w|} = \frac{h(2T - t)}{|w|} \) and output sampled at \( t = 4T \). The resulting implementation is depicted here:
Solution 2.

(a) As is evident from the problem, it’s a case of waveform detection where the signals lie in a vector space of orthonormal basis

\[ \psi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \mathbb{1}_{[0,T]}(t) \]
\[ \psi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t) \mathbb{1}_{[0,T]}(t). \]

The signals can then be represented by vectors in the space spanned by the orthonormal basis \( \psi_1(t) \) and \( \psi_2(t) \) as

\[ c_1 = (\sqrt{E}, \sqrt{E}) \]
\[ c_2 = (-\sqrt{E}, \sqrt{E}) \]
\[ c_3 = (-\sqrt{E}, -\sqrt{E}) \]
\[ c_4 = (\sqrt{E}, -\sqrt{E}). \]

We will use the same receiver structure given in the book for orthonormal bases. Using the matched filters \( \psi_1(T - t) \) and \( \psi_2(T - t) \), the receiver obtains the pair \( Y = (Y_1, Y_2)^T \), where

\[ Y_1 = \langle R(t), \psi_1(t) \rangle, \]
\[ Y_2 = \langle R(t), \psi_2(t) \rangle. \]

As indicated in the book, the MAP decoder chooses the \( i \) that maximizes \( \langle y, c_i \rangle + q_i \), where \( q_i = \frac{1}{2} (N_0 \ln P_H(i) - ||c_i||^2) \). Now since the waveforms are equi-probable and equi-energy, the additive constant terms \( q_i \) are the same for each hypothesis. Therefore, the decoder can choose the \( i \) that maximizes \( \langle y, c_i \rangle \). The decoding regions are therefore

\[ R_1 = \{(Y_1, Y_2) : Y_1 \geq 0, Y_2 \geq 0 \} \]
\[ R_2 = \{(Y_1, Y_2) : Y_1 < 0, Y_2 \geq 0 \} \]
\[ R_3 = \{(Y_1, Y_2) : Y_1 < 0, Y_2 < 0 \} \]
\[ R_4 = \{(Y_1, Y_2) : Y_1 \geq 0, Y_2 < 0 \}. \]

(b) The probability of error is the same for each hypothesis. If \( Z_1 \) and \( Z_2 \) are the
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The receiver then processes the received signal $R(t)$ through the matched filter $h_j(t)$ to obtain $(R * h_j)(t)$. This signal is sampled at time $T$ to yield the value needed for the MAP decision.

(b) We need $m$ matched filters, one for each signal:

\[
\begin{align*}
R(t) & \quad h_0(t) & \quad t = T & \quad Y_0 \\
& \quad h_1(t) & \quad t = T & \quad Y_1 \\
& \quad \vdots & \quad \vdots & \quad \vdots \\
& \quad h_{m-1}(t) & \quad t = T & \quad Y_{m-1}
\end{align*}
\]

Select $\arg \max_i Y_i$

(c) We can use the following MATLAB program to compute the output of the matched filter.

```matlab
T = 1;
Resolution = 10^(-3);
t = 0:Resolution:T;
nj = 5;

wj = sqrt(2/T) * cos ((2*pi*nj*t)/T);
hj = sqrt(2/T) * cos ((2*pi*nj*t)/T);
output = conv(wj, hj);

(h_5 * w_5)(t)
```

Note that the resulting signal is zero for $t \leq 0$ and also for $t \geq 2T$. The figure also reveals why sampling at time $t = T$ is a good idea: the value of the matched filter output signal is maximal.
(d) We first prove that the voltage response of the LC circuit to a current impulse \( i(t) = \delta(t) \) is indeed \( u(t) = \frac{1}{C} \cos \omega_0 t \) for \( \omega_0 = \frac{1}{\sqrt{LC}} \).

We assume that the circuit is at rest (\( u_C = 0 \) and \( i_L = 0 \) for \( t < 0 \)).

The effect of injecting a pulse of current \( i(t) = \delta(t) \) is that of charging the capacitor. (The current in an inductor can not jump.) The result is

\[
\begin{align*}
u_C(t) &= \frac{1}{C} \int i_C(t) \, dt = \frac{1}{C} \int \delta(t) \, dt = \frac{1}{C}.
\end{align*}
\]

So the initial conditions of the circuit we are analyzing are

\[
u_C(0) = \frac{1}{C}, \quad i_L(0) = 0.
\]

From Kirchhoff’s laws, we have

\[
\begin{align*}
u_C(t) &= u_L(t) \\
i_C(t) &= -i_L(t),
\end{align*}
\]

and from the circuit component equations we have

\[
\begin{align*}
i_C(t) &= C \frac{d}{dt} u_C(t) \\
u_L(t) &= L \frac{d}{dt} i_L(t).
\end{align*}
\]

Combining we obtain

\[
i_C(t) = -CL \frac{d^2}{dt^2} i_C(t)
\]

or

\[
\omega_0^2 i_C(t) + \frac{d^2}{dt^2} i_C(t) = 0,
\]

where \( \omega_0^2 = \frac{1}{CL} \). The general solution of this differential equation is

\[
i_C(t) = Ae^{j\omega_0 t} + Be^{-j\omega_0 t}.
\]

From the initial conditions \( i_L(0) = i_C(0) = 0 \) we obtain \( B = -A \). Hence

\[
i_C(t) = A(e^{j\omega_0 t} - e^{-j\omega_0 t}).
\]

Now

\[
\begin{align*}u(t) &= u_L(t) = L \frac{d}{dt} i_L(t) \\
&= -L \frac{d}{dt} i_C(t) \\
&= -LA(j\omega_0 e^{j\omega_0 t} + j\omega_0 e^{-j\omega_0 t}) \\
&= -2LA j\omega_0 \cos \omega_0 t.
\end{align*}
\]
From \( u(0) = \frac{1}{C} \) we obtain \( \frac{1}{C} = -2LAj\omega_0 \), hence

\[
u(t) = \frac{1}{C} \cos \omega_0 t, \quad t \geq 0.
\]

We have proved that the impulse response \( h(t) \) interpreted as the voltage response to the current impulse is

\[
h(t) = \frac{1}{C} \cos \omega_0 t, \quad t \geq 0.
\]

Now let the current at the input of the circuit be \( i(t) = w_j(t) \). Then the voltage \( u(t) \) at the output is

\[
u(t) = (w_j * h)(t),
\]

and it is clear that \( L \) and \( C \) have to be chosen such that \( h(t) \) in the last equation becomes \( h_j(t) \), i.e. such that

\[
\frac{2\pi n_j}{T} = \frac{1}{\sqrt{LC}}, \quad \text{and} \quad \frac{1}{C} = \sqrt{\frac{2}{T}}.
\]

The difference between the circuit and the true matched filter is that the impulse response of the matched filter is limited to the interval \( 0 \leq t \leq T \); the impulse response of an ideal resonance circuit is not time-limited. However, at time \( t = T \), the output of the resonance circuit gives the correct value.

Thus, if we make sure that at time \( t = 0 \), all the energy in \( L \) and \( C \) is dumped, and at time \( T \), we sample \( u(t) \), then we have indeed implemented a matched filter. That is, we need two switches as shown below:

Since this circuit integrates and then dumps, it is called the integrate-and-dump circuit.

**Solution 4.**
(a) The Cauchy-Schwarz inequality states
\[ |\langle x, y \rangle| \leq \|x\| \cdot \|y\| \]
with equality if and only if \( x = \alpha y \) for some scalar \( \alpha \). For our problem, we can write
\[ |\langle w, \phi \rangle|^2 \leq \|w\|^2 \cdot \|\phi\|^2 = \|w\|^2 \]
with equality if and only if \( \phi = \alpha w \) for some scalar \( \alpha \). Thus, the maximizing \( \phi(t) \)
is simply a scaled version of \( w(t) \).

(b) The problem is
\[
\max_{\phi_1, \phi_2} (c_1 \phi_1 + c_2 \phi_2) \text{ subject to } \phi_1^2 + \phi_2^2 = 1.
\]

Thus, we can reduce by setting \( \phi_2 = \sqrt{1 - \phi_1^2} \) to obtain
\[
\max_{\phi_1} \left( c_1 \phi_1 + c_2 \sqrt{1 - \phi_1^2} \right).
\]

This maximum is found by taking the derivative:
\[
\frac{d}{d\phi_1} \left( c_1 \phi_1 + c_2 \sqrt{1 - \phi_1^2} \right) = c_1 - c_2 \frac{\phi_1}{\sqrt{1 - \phi_1^2}}.
\]

Setting this equal to zero yields \( c_1 = c_2 \frac{\phi_1}{\sqrt{1 - \phi_1^2}} \), i.e.
\[
c_1^2 = c_2^2 \frac{\phi_1^2}{1 - \phi_1^2}.
\]

This immediately gives \( \phi_1 = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \) and thus \( \phi_2 = \frac{c_2}{\sqrt{c_1^2 + c_2^2}} \), which are colinear to \( c_1 \) and \( c_2 \) respectively.

(c) Passing an input \( w(t) \) through a filter with impulse response \( h(t) \) generates output waveform \( y(t) = \int w(\tau)h(t-\tau)d\tau \). If this waveform \( y(t) \) is sampled at time \( t = T \), then the output sample is
\[
y(T) = \int w(\tau)h(T-\tau) \, d\tau. \tag{1}
\]

An example signal \( w(\tau) \) is shown in Figure (a) below. The filter is then the waveform shown in Figure (b), and the convolution term of the filter in Figure (c). Finally, the filter term \( h(T - \tau) \) of Equation (1) is shown in Figure (d). One can see that \( h(T - \tau) = w(\tau) \), so indeed
\[
y(T) = \int w(\tau)h(T-\tau) \, d\tau = \int w^2(\tau) \, d\tau = \int_0^T w^2(\tau) \, d\tau.
\]
Solution 5.

(a) The third component of $c_i$ is zero for all $i$. Furthermore $Z_1$, $Z_2$ and $Z_3$ are zero mean i.i.d. Gaussian random variables. Hence,

$$f_{Y|H}(y|i) = f_{Z_1}(y_1 - c_{i,1})f_{Z_2}(y_2 - c_{i,2})f_{Z_3}(y_3),$$

which is in the form $g_i(T(y))h(y)$ for $T(y) = (y_1, y_2)^T$ and $h(y) = f_{Z_3}(y_3)$. Hence, by the Fisher-Neyman factorization theorem, $T(Y) = (Y_1, Y_2)^T$ is a sufficient statistic.

(b) We have $Y_3 = Z_3 = Z_2$. By observing $Y_3$, we can remove the noise in the second component of $Y$. Specifically, we have $c_{i,2} = Y_2 - Y_3$. If the second component is different for each hypothesis, then the receiver can make an error-free decision which is not possible to do using only $(Y_1, Y_2)$ (see the next question for more on this). We can see that $Y_3$ contains very useful information and can’t be discarded. Therefore, $(Y_1, Y_2)$ is not a sufficient statistic.

(c) If we have only $(Y_1, Y_2)$ then the hypothesis testing problem will be

$$H = i : (Y_1, Y_2) = (c_{i,1}, c_{i,2}) + (Z_1, Z_2) \quad i = 0, 1.$$

Using the fact that $c_0 = (1, 0, 0)^T$ and $c_1 = (0, 1, 0)^T$, the ML test becomes

$$y_1 - y_2 \stackrel{H=0}{\gtrless} 0.$$

Under $H = 0$, $Y_1 - Y_2$ is a Gaussian random variable with mean 1 and variance $2\sigma^2$, and so $P_e(0) = Q\left(\frac{1}{\sqrt{2\sigma^2}}\right)$. By symmetry $P_e(1) = Q\left(\frac{1}{\sqrt{2\sigma^2}}\right)$, and so the probability of the error will be $P_e = \frac{1}{2}(P_e(0) + P_e(1)) = Q\left(\frac{1}{\sqrt{2\sigma^2}}\right)$.

Now assume that we have access to $Y_1$, $Y_2$ and $Y_3$. $Y_3$ contains $Z_3 = Z_2$ under both hypotheses. Hence, $Y_2 - Y_3 = c_{i,2} + Z_2 - Z_3 = c_{i,2}$. This shows that at the receiver
we can observe the second component of \( c_i \) without noise. As the second component is different under both hypotheses, we can make an error-free decision about \( H \) and the decision rule will be:

\[
\hat{H} = \begin{cases} 
0 & y_2 - y_3 = 0 \\
1 & y_2 - y_3 = 1
\end{cases}
\]

Clearly this decision rule minimizes the probability of the error. We see that \( Y_3 \) allows us to reduce the probability of the error; this shows once again that \((Y_1, Y_2)\) can’t be a sufficient statistic.

**Solution 6.**

(a) The optimal solution is to pass \( R(t) \) through the matched filter \( w(T - t) \) and sample the result at \( t = T \) to get a sufficient statistic denoted by \( Y \) (In this problem, \( T = 1 \)). Note that \( Y = S + N \), where \( S \) and \( N \) are random variables denoting the signal and the noise components respectively. Under \( H = i \), \( Y \sim \mathcal{N}(\alpha_i, N_0/2) \), where \( \alpha_0, \ldots, \alpha_3 \) are \( 3c, c, -c \) and \( -3c \) respectively.

Let \( \hat{X} \) be the recovered signal value at the receiver. Based on the nearest neighbor decision rule, the receiver chooses the value of \( \hat{X} \) in the following fashion:

\[
\hat{X} = \begin{cases} 
+3 & Y \in [2c, \infty) \\
+1 & Y \in [0, 2c) \\
-1 & Y \in [-2c, 0) \\
-3 & Y \in (-\infty, -2c). 
\end{cases}
\]

(b) The probability of error is given by

\[
P_e = \sum_{i=0}^{3} \frac{1}{4} \Pr \{ \text{error} | H = i \} = \frac{1}{4} \left[ Q \left( \frac{c/4}{\sqrt{N_0/2}} \right) + 2Q \left( \frac{c}{\sqrt{N_0/2}} \right) + 2Q \left( \frac{3c/4}{\sqrt{N_0/2}} \right) + Q \left( \frac{c/4}{\sqrt{N_0/2}} \right) \right] = \frac{3}{2} Q \left( \frac{c}{\sqrt{N_0/2}} \right).
\]

(c) In this case under \( H = i \), \( Y \sim \mathcal{N}(\alpha_i, N_0/2) \), where \( \alpha_0, \ldots, \alpha_3 \) are \( \frac{9c}{4}, \frac{3c}{4}, -\frac{3c}{4} \) and \( -\frac{9c}{4} \) respectively. Using the decision rule in (2), the probability of error is given by

\[
P_e = \sum_{i=0}^{3} \frac{1}{4} \Pr \{ \text{error} | H = i \} = \frac{1}{4} \left[ Q \left( \frac{c/4}{\sqrt{N_0/2}} \right) + Q \left( \frac{5c/4}{\sqrt{N_0/2}} \right) + Q \left( \frac{3c/4}{\sqrt{N_0/2}} \right) + Q \left( \frac{3c/4}{\sqrt{N_0/2}} \right) \right].
\]

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(d) The noise process \( N(t) \) is a stationary Gaussian random process. So the noise component \( N \) (which is the sample of match-filter output at time \( T \)) is a Gaussian random variable with mean
\[
E[N] = E \left[ \int_{-\infty}^{\infty} N(t)w(t)dt \right] = E \left[ \int_{0}^{1} N(t)dt \right] = 0.
\]
Because the process \( N(t) \) is stationary, without loss of generality we choose the boundaries of the integral to be 0 and \( T \) where in this problem \( T = 1 \).

Now, let us calculate the noise variance.
\[
\text{var}(N) = E[N^2] - E[N]^2 = E[N^2]
\]
\[
= E \left[ \int_{-\infty}^{\infty} N(t)w(t)dt \times \int_{-\infty}^{\infty} N(v)w(v)dv \right]
\]
\[
= E \left[ \int_{0}^{1} N(t)dt \times \int_{0}^{1} N(v)dv \right]
\]
\[
= E \left[ \int_{0}^{1} \int_{0}^{1} N(t)N(v) dt \ dv \right]
\]
\[
= \int_{0}^{1} \int_{0}^{1} K_N(t-v) \ dt \ dv
\]
\[
= \int_{0}^{1} \int_{0}^{1} \frac{1}{4\alpha} e^{-|t-v|/\alpha} \ dt \ dv
\]
\[
= \frac{1}{2} \left( \alpha \left( e^{-1/\alpha} - 1 \right) + 1 \right).
\]

Thus the new probability of error is given by
\[
P_e = \sum_{i=0}^{3} \frac{1}{4} \Pr \{\text{error} | H = i\}
\]
\[
= \frac{1}{4} \left[ Q \left( \frac{c}{\sqrt{\frac{1}{2} \left( \alpha \left( e^{-1/\alpha} - 1 \right) + 1 \right)}} \right) + 2Q \left( \frac{c}{\sqrt{\frac{1}{2} \left( \alpha \left( e^{-1/\alpha} - 1 \right) + 1 \right)}} \right)
\]
\[
+ 2Q \left( \frac{c}{\sqrt{\frac{1}{2} \left( \alpha \left( e^{-1/\alpha} - 1 \right) + 1 \right)}} \right) + Q \left( \frac{c}{\sqrt{\frac{1}{2} \left( \alpha \left( e^{-1/\alpha} - 1 \right) + 1 \right)}} \right) \right]
\]
\[
= \frac{3}{2} Q \left( \frac{c}{\sqrt{\frac{1}{2} \left( \alpha \left( e^{-1/\alpha} - 1 \right) + 1 \right)}} \right).
\]