ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 13 Principles of Digital Communications

Solutions to Problem Set 6 Mar. 31, 2015

Solution 1.

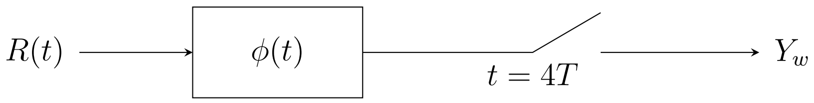
(a) The signals that are being sent are $w_0(t)$ and $w_1(t)$, where

$$w_0(t) = w(t),$$
$$w_1(t) = 0.$$

Clearly the span of $\mathcal{W} = \{w_0, w_1\}$ is one dimensional, and it is spanned by w. Therefore, we can take $\{\psi\}$ as an orthonormal basis for $span(\mathcal{W})$, where $\psi(t) = \frac{w(t)}{||w||}$. The codewords corresponding to w_0 and w_1 are:

- $c_0 = ||w|| \in \mathbb{R}$.
- $c_1 = 0 \in \mathbb{R}$.

The maximum likelihood receiver for the observable $R(t)$ uses the matched filter with impulse response $\phi(t) = \psi(4T - t) = \frac{w(4T-t)}{\|w\|}$. The n-tuple former computes $Y_w = \langle R(t), \psi(t) \rangle$ using the matched filter $\phi(t)$ as shown below:



The maximum likelihood receiver decides $\hat{H} = 0$ if $Y_w > \frac{\|w\|}{2}$ and $\hat{H} = 1$ otherwise.

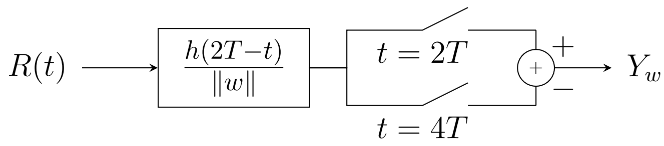
(b) The probability of error can be directly inferred to be

$$P_e = Q\left(\frac{\|w\|}{2\sigma}\right) = Q\left(\frac{\|w\|}{2\sqrt{\frac{N_0}{2}}}\right) = Q\left(\frac{\|w\|}{\sqrt{2N_0}}\right).$$

(c) The question is how to compute $\langle R(t), \frac{w(t)}{||w||} \rangle$ using $h(t)$ instead of $w(t)$. Notice that we have $w(t) = h(t) - h(t - 2T)$. Therefore,

$$\langle R(t), \frac{w(t)}{||w||} \rangle = \langle R(t), \frac{h(t)}{||w||} \rangle - \langle R(t), \frac{h(t-2T)}{||w||} \rangle.$$

The first term can be obtained via a filter of impulse response $\frac{h(2T-t)}{||w||}$ and output sampled at $t = 2T$. The second term can be obtained via a filter of impulse response $\frac{h((4T-t)-2T)}{||w||} = \frac{h(2T-t)}{||w||}$ and output sampled at $t = 4T$. The resulting implementation is depicted here:



SOLUTION 2.

(a) As is evident from the problem, it's a case of waveform detection where the signals lie in a vector space of orthonormal basis

$$\psi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\mathbb{1}_{[0,T]}(t)$$
$$\psi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t)\mathbb{1}_{[0,T]}(t).$$

The signals can then be represented by vectors in the space spanned by the orthonormal basis $\psi_1(t)$ and $\psi_2(t)$ as

$$c_1 = (\sqrt{\mathcal{E}}, \sqrt{\mathcal{E}})$$
$$c_2 = (-\sqrt{\mathcal{E}}, \sqrt{\mathcal{E}})$$
$$c_3 = (-\sqrt{\mathcal{E}}, -\sqrt{\mathcal{E}})$$
$$c_4 = (\sqrt{\mathcal{E}}, -\sqrt{\mathcal{E}}).$$

We will use the same receiver structure given in the book for orthonormal bases. Using the matched filters $\psi_1(T-t)$ and $\psi_2(T-t)$, the receiver obtains the pair $Y = (Y_1, Y_2)^\mathsf{T}$, where

$$Y_1 = \langle R(t), \psi_1(t)\rangle,$$
$$Y_2 = \langle R(t), \psi_2(t)\rangle.$$

As indicated in the book, the MAP decoder chooses the i that maximizes $\langle y, c_i\rangle + q_i$, where $q_i = \frac{1}{2}(N_0 \ln P_H(i) - ||c_i||^2)$. Now since the waveforms are equi-probable and equi-energy, the additive constant terms q_i are the same for each hypothesis. Therefore, the decoder can choose the i that maximizes $\langle y, c_i\rangle$. The decoding regions are therefore

$$\mathcal{R}_1 = \{(Y_1, Y_2) : Y_1 \geq 0, Y_2 \geq 0\}$$
$$\mathcal{R}_2 = \{(Y_1, Y_2) : Y_1 < 0, Y_2 \geq 0\}$$
$$\mathcal{R}_3 = \{(Y_1, Y_2) : Y_1 < 0, Y_2 < 0\}$$
$$\mathcal{R}_4 = \{(Y_1, Y_2) : Y_1 \geq 0, Y_2 < 0\}.$$

(b) The probability of error is the same for each hypothesis. If Z_1 and Z_2 are the projections of the noise onto $\psi_1(t)$ and $\psi_2(t)$, respectively, then

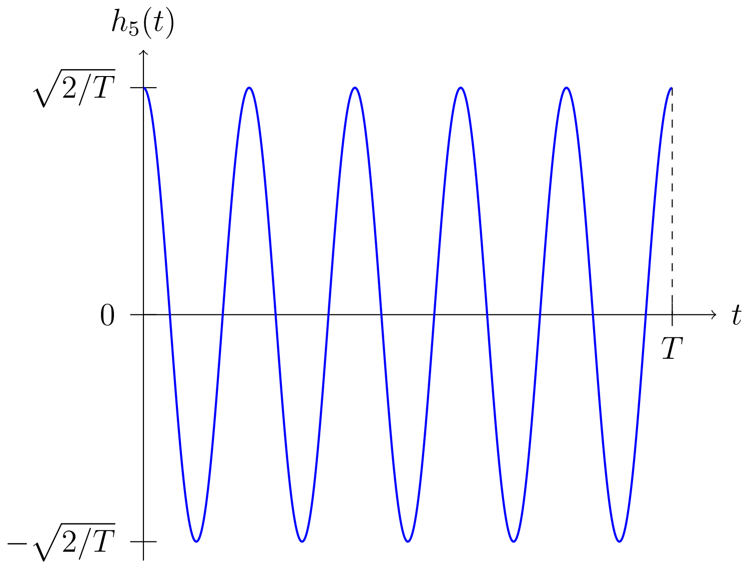
$$\begin{aligned}
P_e &= 1 - \Pr\left\{ Z_1 \geq -\frac{\sqrt{\mathcal{E}}}{\sigma}, Z_2 \geq -\frac{\sqrt{\mathcal{E}}}{\sigma} \right\} \\
&= 1 - \left[Q\left(-\frac{\sqrt{\mathcal{E}}}{\sqrt{\frac{N_0}{2}}} \right) \right]^2 \\
&= 1 - \left[Q\left(-\sqrt{\frac{2\mathcal{E}}{N_0}} \right) \right]^2 \\
&= 1 - \left[1 - Q\left(\sqrt{\frac{2\mathcal{E}}{N_0}} \right) \right]^2 \\
&= 2Q\left(\sqrt{\frac{2\mathcal{E}}{N_0}} \right) - Q^2\left(\sqrt{\frac{2\mathcal{E}}{N_0}} \right).
\end{aligned}$$

Solution 3.

(a) The matched filter is the filter whose impulse response is a delayed, time-reversed version of a signal $w_j(t)$, i.e.

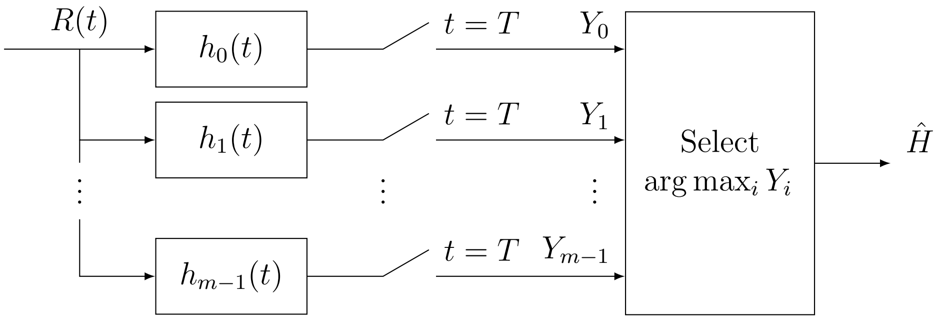
$$\begin{aligned}
h_j(t) = w_j(T-t) &= \sqrt{\frac{2}{T}} \cos \frac{2\pi n_j (T-t)}{T} \mathbb{1}_{[0,T]}(t) \\
&= \sqrt{\frac{2}{T}} \cos \frac{2\pi n_j t}{T} \mathbb{1}_{[0,T]}(t).
\end{aligned}$$

As an example, $h_5(t)$ is shown here:



The receiver then processes the received signal $R(t)$ through the matched filter $h_j(t)$ to obtain $(R * h_j)(t)$. This signal is sampled at time T to yield the value needed for the MAP decision.

(b) We need m matched filters, one for each signal:

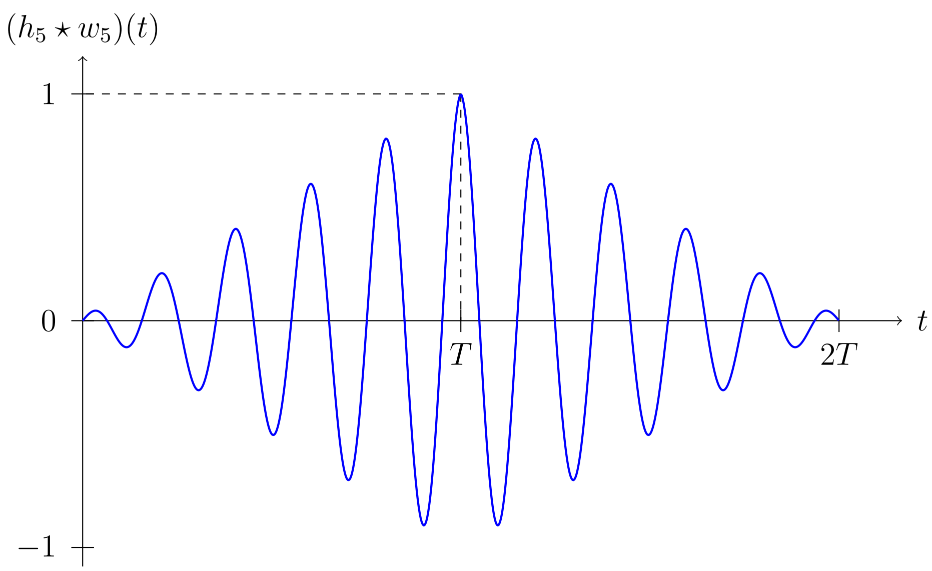


(c) We can use the following MATLAB program to compute the output of the matched filter.

```
T = 1;
Resolution = 10^(-3);
t = 0:Resolution:T;
nj = 5;

wj = sqrt(2/T) * cos ( (2*pi*nj*t)/T );
hj = sqrt(2/T) * cos ( (2*pi*nj*t)/T );

output = conv(wj, hj);
```



Note that the resulting signal is *zero* for $t \leq 0$ and also for $t \geq 2T$. The figure also reveals why sampling at time $t = T$ is a good idea: the value of the matched filter output signal is maximal.

(d) We first prove that the voltage response of the LC circuit to a current impulse $i(t) = \delta(t)$ is indeed $u(t) = \frac{1}{C} \cos \omega_0 t$ for $\omega_0 = \frac{1}{\sqrt{LC}}$.

We assume that the circuit is at rest ($u_C = 0$ and $i_L = 0$ for $t < 0$).

The effect of injecting a pulse of current $i(t) = \delta(t)$ is that of charging the capacitor. (The current in an inductor can not jump.) The result is

$$u_C(t) = \frac{1}{C} \int i_C(t)\ dt = \frac{1}{C} \int \delta(t)\ dt = \frac{1}{C}.$$

So the initial conditions of the circuit we are analyzing are

$$u_C(0) = \frac{1}{C} \qquad i_L(0) = 0.$$

From Kirchhoff's laws, we have

$$u_C(t) = u_L(t)$$

$$i_C(t) = -i_L(t),$$

and from the circuit component equations we have

$$i_C(t) = C \frac{d}{dt} u_C(t)$$

$$u_L(t) = L \frac{d}{dt} i_L(t).$$

Combining we obtain

$$i_C(t) = -CL \frac{d^2}{dt^2} i_C(t)$$

or

$$\omega_0^2 i_C(t) + \frac{d^2}{dt^2} i_C(t) = 0,$$

where $\omega_0^2 = \frac{1}{CL}$. The general solution of this differential equation is

$$i_C(t) = Ae^{\mathrm{j}\omega_0 t} + Be^{-\mathrm{j}\omega_0 t}.$$

From the initial conditions $i_L(0) = i_C(0) = 0$ we obtain $B = -A$. Hence

$$i_C(t) = A(e^{\mathrm{j}\omega_0 t} - e^{-\mathrm{j}\omega_0 t}).$$

Now

$$\begin{aligned} u(t) = u_L(t) &= L \frac{d}{dt} i_L(t) \\ &= -L \frac{d}{dt} i_C(t) \\ &= -LA(\mathrm{j}\omega_0 e^{\mathrm{j}\omega_0 t} + \mathrm{j}\omega_0 e^{-\mathrm{j}\omega_0 t}) \\ &= -2LA\mathrm{j}\omega_0 \cos \omega_0 t\ . \end{aligned}$$

From $u(0) = \frac{1}{C}$ we obtain $\frac{1}{C} = -2LA\mathrm{j}\omega_0$, hence

$$u(t) = \frac{1}{C}\cos\omega_0 t, \qquad t \geq 0.$$

We have proved that the impulse response $h(t)$ interpreted as the voltage response to the current impulse is

$$h(t) = \frac{1}{C}\cos\omega_0 t, \qquad t \geq 0.$$

Now let the current at the input of the circuit be $i(t) = w_j(t)$. Then the voltage $u(t)$ at the output is

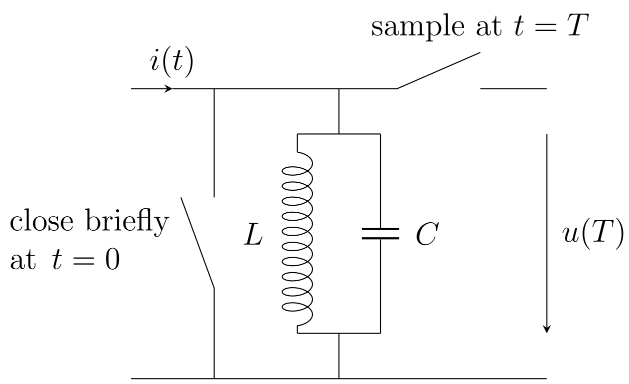
$$u(t) = (w_j * h)(t),$$

and it is clear that L and C have to be chosen such that $h(t)$ in the last equation becomes $h_j(t)$, i.e. such that

$$\begin{aligned} \frac{2\pi n_j}{T} &= \frac{1}{\sqrt{LC}}, \text{ and} \\ \frac{1}{C} &= \sqrt{\frac{2}{T}}. \end{aligned}$$

The difference between the circuit and the true matched filter is that the impulse response of the matched filter is limited to the interval $0 \leq t \leq T$; the impulse response of an ideal resonance circuit is *not* time-limited. However, at time $t = T$, the output of the resonance circuit gives the correct value.

Thus, if we make sure that at time $t = 0$, all the energy in L and C is *dumped*, and at time T, we sample $u(t)$, then we have indeed implemented a matched filter. That is, we need two switches as shown below:



Since this circuit integrates and then dumps, it is called the *integrate-and-dump* circuit.

SOLUTION 4.

(a) The Cauchy-Schwarz inequality states

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$$

with equality if and only if $x = \alpha y$ for some scalar α. For our problem, we can write

$$|\langle w, \phi\rangle|^2 \leq \|w\|^2 \cdot \|\phi\|^2 = \|w\|^2$$

with equality if and only if $\phi = \alpha w$ for some scalar α. Thus, the maximizing $\phi(t)$ is simply a scaled version of $w(t)$.

(b) The problem is

$$\max_{\phi_1, \phi_2} (c_1\phi_1 + c_2\phi_2) \text{ subject to } \phi_1^2 + \phi_2^2 = 1.$$

Thus, we can reduce by setting $\phi_2 = \sqrt{1-\phi_1^2}$ to obtain

$$\max_{\phi_1} \left(c_1\phi_1 + c_2\sqrt{1-\phi_1^2} \right).$$

This maximum is found by taking the derivative:

$$\frac{d}{d\phi_1} \left(c_1\phi_1 + c_2\sqrt{1-\phi_1^2} \right) = c_1 - c_2\frac{\phi_1}{\sqrt{1-\phi_1^2}}.$$

Setting this equal to zero yields $c_1 = c_2\frac{\phi_1}{\sqrt{1-\phi_1^2}}$, i.e.

$$c_1^2 = c_2^2\frac{\phi_1^2}{1-\phi_1^2}.$$

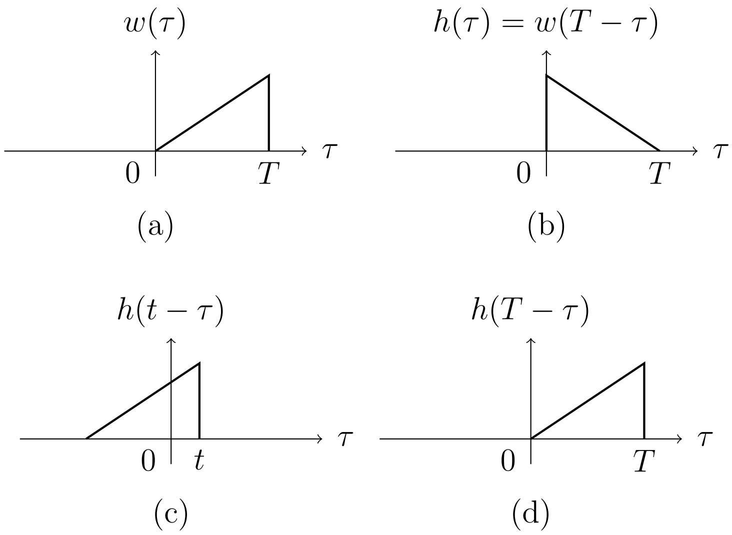
This immediately gives $\phi_1 = \frac{c_1}{\sqrt{c_1^2+c_2^2}}$ and thus $\phi_2 = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$, which are colinear to c_1 and c_2 respectively.

(c) Passing an input $w(t)$ through a filter with impulse response $h(t)$ generates output waveform $y(t) = \int w(\tau)h(t-\tau)d\tau$. If this waveform $y(t)$ is sampled at time $t = T$, then the output sample is

$$y(T) = \int w(\tau)h(T-\tau)\, d\tau. \tag{1}$$

An example signal $w(\tau)$ is shown in Figure (a) below. The filter is then the waveform shown in Figure (b), and the convolution term of the filter in Figure (c). Finally, the filter term $h(T-\tau)$ of Equation (1) is shown in Figure (d). One can see that $h(T-\tau) = w(\tau)$, so indeed

$$y(T) = \int w(\tau)h(T-\tau)\, d\tau = \int w^2(\tau)\, d\tau = \int_0^T w^2(\tau)\, d\tau.$$



SOLUTION 5.

(a) The third component of c_i is zero for all i. Furthermore Z_1, Z_2 and Z_3 are zero mean i.i.d. Gaussian random variables. Hence,

$$f_{Y|H}(y|i) = f_{Z_1}(y_1 - c_{i,1}) f_{Z_2}(y_2 - c_{i,2}) f_{Z_3}(y_3),$$

which is in the form $g_i(T(y))h(y)$ for $T(y) = (y_1, y_2)^T$ and $h(y) = f_{Z_3}(y_3)$. Hence, by the Fisher-Neyman factorization theorem, $T(Y) = (Y_1, Y_2)^T$ is a sufficient statistic.

(b) We have $Y_3 = Z_3 = Z_2$. By observing Y_3, we can remove the noise in the second component of Y. Specifically, we have $c_{i,2} = Y_2 - Y_3$. If the second component is different for each hypothesis, then the receiver can make an error-free decision which is not possible to do using only (Y_1, Y_2) (see the next question for more on this). We can see that Y_3 contains very useful information and can't be discarded. Therefore, (Y_1, Y_2) is not a sufficient statistic.

(c) If we have only (Y_1, Y_2) then the hypothesis testing problem will be

$$H = i : (Y_1, Y_2) = (c_{i,1}, c_{i,2}) + (Z_1, Z_2) \quad i = 0, 1.$$

Using the fact that $c_0 = (1, 0, 0)^\mathsf{T}$ and $c_1 = (0, 1, 0)^\mathsf{T}$, the ML test becomes

$$y_1 - y_2 \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0.$$

Under $H = 0$, $Y_1 - Y_2$ is a Gaussian random variable with mean 1 and variance $2\sigma^2$, and so $P_e(0) = Q(\frac{1}{\sqrt{2}\sigma})$. By symmetry $P_e(1) = Q(\frac{1}{\sqrt{2}\sigma})$, and so the probability of the error will be $P_e = \frac{1}{2}(P_e(0) + P_e(1)) = Q(\frac{1}{\sqrt{2}\sigma})$.

Now assume that we have access to Y_1, Y_2 and Y_3. Y_3 contains $Z_3 = Z_2$ under both hypotheses. Hence, $Y_2 - Y_3 = c_{i,2} + Z_2 - Z_3 = c_{i,2}$. This shows that at the receiver we can observe the second component of c_i without noise. As the second component is different under both hypotheses, we can make an error-free decision about H and the decision rule will be:

$$\hat{H} = \begin{cases} 0 & y_2 - y_3 = 0 \\ 1 & y_2 - y_3 = 1 \end{cases}$$

Clearly this decision rule minimizes the probability of the error. We see that Y_3 allows us to reduce the probability of the error; this shows once again that (Y_1, Y_2) can't be a sufficient statistic.

SOLUTION 6.

(a) The optimal solution is to pass $R(t)$ through the matched filter $w(T-t)$ and sample the result at $t = T$ to get a sufficent statistic denoted by Y (In this problem, $T = 1$). Note that $Y = S + N$, where S and N are random variables denoting the signal and the noise components respectively. Under $H = i$, $Y \sim \mathcal{N}(\alpha_i, N_0/2)$, where $\alpha_0, \ldots, \alpha_3$ are $3c$, c, $-c$ and $-3c$ respectively.

Let $\hat{X}$ be the recovered signal value at the receiver. Based on the nearest neighbor decision rule, the receiver chooses the value of $\hat{X}$ in the following fashion:

$$\hat{X} = \begin{cases} +3, & Y \in [2c, \infty) \\ +1, & Y \in [0, 2c) \\ -1, & Y \in [-2c, 0) \\ -3, & Y \in [-\infty, -2c). \end{cases} \tag{2}$$

(b) The probability of error is given by

$$\begin{aligned} P_e &= \sum_{i=0}^{3} \frac{1}{4} \Pr\{\text{error}|H = i\} \\ &= \frac{1}{4}\left[Q\left(\frac{c}{\sqrt{N_0/2}}\right) + 2Q\left(\frac{c}{\sqrt{N_0/2}}\right) + 2Q\left(\frac{c}{\sqrt{N_0/2}}\right) + Q\left(\frac{c}{\sqrt{N_0/2}}\right)\right] \\ &= \frac{3}{2}Q\left(\frac{c}{\sqrt{N_0/2}}\right). \end{aligned}$$

(c) In this case under $H = i$, $Y \sim \mathcal{N}(\alpha_i, N_0/2)$, where $\alpha_0, \ldots, \alpha_3$ are $\frac{9c}{4}$, $\frac{3c}{4}$, $\frac{-3c}{4}$ and $\frac{-9c}{4}$ respectively. Using the decision rule in (2), the probability of error is given by

$$\begin{aligned} P_e &= \sum_{i=0}^{3} \frac{1}{4} \Pr\{\text{error}|H = i\} \\ &= \frac{1}{4}\left[Q\left(\frac{c/4}{\sqrt{N_0/2}}\right) + Q\left(\frac{5c/4}{\sqrt{N_0/2}}\right) + Q\left(\frac{3c/4}{\sqrt{N_0/2}}\right)\right. \\ &\quad \left. + Q\left(\frac{5c/4}{\sqrt{N_0/2}}\right) + Q\left(\frac{3c/4}{\sqrt{N_0/2}}\right) + Q\left(\frac{c/4}{\sqrt{N_0/2}}\right)\right]. \end{aligned}$$

(d) The noise process $N(t)$ is a stationary Gaussian random process. So the noise component N (which is the sample of match-filter output at time T) is a Gaussian random variable with mean

$$\mathbb{E}\left[N\right] = \mathbb{E}\left[\int_{-\infty}^{\infty} N(t)w(t)dt\right] = \mathbb{E}\left[\int_{0}^{1} N(t)dt\right] = 0.$$

Because the process $N(t)$ is stationary, without loss of generality we choose the boundaries of the integral to be 0 and T where in this problem $T = 1$.

Now, let us calculate the noise variance.

$$\begin{aligned}
\mathrm{var}(N) &= \mathbb{E}\left[N^2\right] - \mathbb{E}\left[N\right]^2 = \mathbb{E}\left[N^2\right] \\
&= \mathbb{E}\left[\int_{-\infty}^{\infty} N(t)w(t)dt \times \int_{-\infty}^{\infty} N(v)w(v)dv\right] \\
&= \mathbb{E}\left[\int_{0}^{1} N(t)dt \times \int_{0}^{1} N(v)dv\right] \\
&= \mathbb{E}\left[\int_{0}^{1}\int_{0}^{1} N(t)N(v)\ dt\ dv\right] \\
&= \int_{0}^{1}\int_{0}^{1} K_N(t-v)\ dt\ dv \\
&= \int_{0}^{1}\int_{0}^{1} \frac{1}{4\alpha}e^{-|t-v|/\alpha}\ dt\ dv \\
&= \frac{1}{2}\left(\alpha\left(e^{-1/\alpha} - 1\right) + 1\right).
\end{aligned}$$

Thus the new probability of error is given by

$$\begin{aligned}
P_e &= \sum_{i=0}^{3} \frac{1}{4}\Pr\left\{\text{error}|H = i\right\} \\
&= \frac{1}{4}\left[Q\left(\frac{c}{\sqrt{\frac{1}{2}\left(\alpha\left(e^{-1/\alpha} - 1\right) + 1\right)}}\right) + 2Q\left(\frac{c}{\sqrt{\frac{1}{2}\left(\alpha\left(e^{-1/\alpha} - 1\right) + 1\right)}}\right)\right. \\
&\quad \left. + 2Q\left(\frac{c}{\frac{1}{2}\left(\alpha\left(e^{-1/\alpha} - 1\right) + 1\right)}\right) + Q\left(\frac{c}{\sqrt{\frac{1}{2}\left(\alpha\left(e^{-1/\alpha} - 1\right) + 1\right)}}\right)\right] \\
&= \frac{3}{2}Q\left(\frac{c}{\sqrt{\frac{1}{2}\left(\alpha\left(e^{-1/\alpha} - 1\right) + 1\right)}}\right).
\end{aligned}$$