Handout 11 Principles of Digital Communications
Solutions to Problem Set 5 Mar. 24, 2015

Solution 1.

(a) Let the two hypotheses be $H = 0$ and $H = 1$ when c_0 and c_1 are transmitted, respectively. The ML decision rule is

$$f_{Y_1 Y_2|H}(y_1, y_2|1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} f_{Y_1 Y_2|H}(y_1, y_2|0).$$

Because Z_1 and Z_2 are independent, we can write

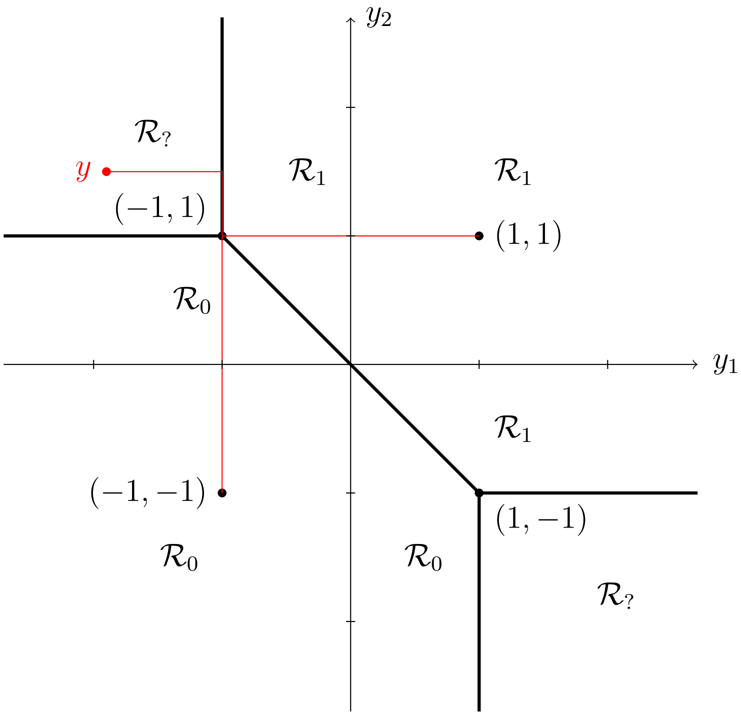
$$\frac{1}{2}e^{-|y_1-1|}\frac{1}{2}e^{-|y_2-1|} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{1}{2}e^{-|y_1+1|}\frac{1}{2}e^{-|y_2+1|},$$

and, after taking the logarithm,

$$|y_1+1| + |y_2+1| \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} |y_1-1| + |y_2-1|.$$

(b) Because the hypotheses are equally likely and Z_1 and Z_2 have the same distribution, the decision region for $\hat{H} = 0$ contains the points closer to $(-1,-1)$ and the decision region for $\hat{H} = 1$ contains the points closer to $(1,1)$. For this problem, the distance between the points (y_{11}, y_{12}) and (y_{21}, y_{22}) is the Manhattan distance, $|y_{11} - y_{21}| + |y_{12} - y_{22}|$, and not the Euclidean distance.

Let us first consider the points above the line $y_2 = -y_1$. It is easy to notice that the points in the positive quadrant are closer to $(1,1)$ than to $(-1,-1)$, therefore they belong to $\mathcal{R}_1$ $(\hat{H} = 1)$. This is also true if $\{(y_1 \geq 0) \cap (y_2 \in (-1,0))\}$, or if $\{(y_2 \geq 0) \cap (y_1 \in (-1,0))\}$.



Similar reasoning can be applied to the points below the diagonal to determine $\mathcal{R}_0$. The points for which $\{(y_1 \leq -1) \cap (y_2 \geq 1)\}$ or $\{(y_1 \geq 1) \cap (y_2 \leq -1)\}$ are equally distanced to $(-1,-1)$ and $(1,1)$, therefore they can belong to either $\mathcal{R}_0$ or $\mathcal{R}_1$ with the same probability. This region is named $\mathcal{R}_?$.

(c) The two hypotheses are equally probable for the region $\mathcal{R}_?$. Therefore, we can split this region in any way between the decision regions and have the same error probability. Because $\mathcal{R}_1$ is included in the region for which $y_2 > -y_1$ and $\mathcal{R}_0$ does not intersect the region for which $y_2 > -y_1$, the error probability is minimized by deciding $\hat{H} = 1$ if $(y_1 + y_2) > 0$.

(d)

$$
\begin{aligned}
P_e(0) &= \Pr\{Y_1 + Y_2 > 0 | H = 0\} \\
&= \Pr\{Z_1 + Z_2 - 2 > 0\} \\
&= \int_2^\infty \frac{e^{-w}}{4}(1+w)\, dw \\
&= \frac{-e^{-w}}{4}(w+2)\Big|_2^\infty = e^{-2}.
\end{aligned}
$$

By symmetry, and considering that the messages are equally likely, $P_e(0) = P_e(1) = P_e$.

Solution 2. We start by normalizing β_1:

$$\begin{aligned} \|\beta_1\| &= \sqrt{\langle \beta_1, \beta_1 \rangle} = \sqrt{3} \\ \psi_1 &= \frac{\beta_1}{\|\beta_1\|} = (\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}). \end{aligned}$$

We get the next basis vectors as follows:

$$\begin{aligned} \langle \psi_1, \beta_2 \rangle &= \sqrt{3} \\ \phi_2 &= \beta_2 - \sqrt{3}\psi_1 = (1, 1, -1, 0) \\ \|\phi_2\| &= \sqrt{3} \\ \psi_2 &= \frac{\phi_2}{\|\phi_2\|} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0). \end{aligned}$$

We compute

$$\begin{aligned} \langle \psi_1, \beta_3 \rangle &= 0 \\ \langle \psi_2, \beta_3 \rangle &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} \phi_3 &= \beta_3 - 0\psi_1 - 0\psi_2 = (1, 0, 1, -2) \\ \|\phi_3\| &= \sqrt{1+1+4} = \sqrt{6} \\ \psi_3 &= \frac{\phi_3}{\|\phi_3\|} = (\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}). \end{aligned}$$

We proceed similarly to obtain ϕ_4:

$$\begin{aligned} \langle \psi_1, \beta_4 \rangle &= \sqrt{3} \\ \langle \psi_2, \beta_4 \rangle &= 0 \\ \langle \psi_3, \beta_4 \rangle &= \sqrt{6} \\ \phi_4 &= \beta_4 - \sqrt{3}\psi_1 - 0\psi_2 - \sqrt{6}\psi_3 = (0, 0, 0, 0). \end{aligned}$$

As can be seen, the last vector is zero. This shows that the dimensionality of the space spanned by $\beta_1, \cdots, \beta_4$ is only 3, not 4. So the other benefit of Gram-Schmidt orthogonalization is that it gives us the dimension of the space spanned by the initial vectors.

Solution 3.

(a) We use the Gram-Schmidt procedure:

(i) The first step is to normalize the function $\beta_0(t)$, i.e. the first function of the basis that we are looking for is

$$\psi_0(t) = \frac{\beta_0(t)}{\|\beta_0(t)\|} = \frac{\beta_0(t)}{\sqrt{\int \beta_0(t)^2 \, dt}}$$

$$= \frac{\beta_0(t)}{\sqrt{\int_0^1 4t^2 \, dt}} = \frac{\sqrt{3}}{2}\beta_0(t) = \begin{cases} 0, & \text{if } t < 0 \\ \sqrt{3}t, & \text{if } 0 \le t \le 1 \\ 0, & \text{if } t > 1 \end{cases}.$$

(ii) Next, we subtract from $\beta_1(t)$ the components that are in the span of the currently established part of the basis, i.e. in the span of $\{\psi_0(t)\}$. This can be achieved by projecting $\beta_1(t)$ onto $\psi_0(t)$ and then subtracting this projection from $\beta_1(t)$, i.e.

$$\begin{aligned}
\alpha_1(t) &= \beta_1(t) - \langle \beta_1(t), \psi_0(t) \rangle \psi_0(t) = \beta_1(t) - \left(\int \beta_1(t)\psi_0(t) \, dt \right) \psi_0(t) \\
&= \beta_1(t) - \left(\frac{\sqrt{3}}{2} \right) \left(\frac{4}{3} \right) \psi_0(t) \\
&= \beta_1(t) - \frac{2}{\sqrt{3}}\psi_0(t) \\
&= \beta_1(t) - \beta_0(t).
\end{aligned}$$

From this, we find the second basis element as

$$\psi_1(t) = \frac{\alpha_1(t)}{||\alpha_1(t)||} = \begin{cases} 0, & \text{if } t < 1 \\ -\sqrt{3}(t-2), & \text{if } 1 \le t \le 2 \\ 0, & \text{if } t > 2 \end{cases}.$$

(iii) Again, we subtract from $\beta_2(t)$ the components that are in the span of the currently established part of the basis, i.e. in the span of $\{\psi_0(t), \psi_1(t)\}$. This can be achieved by projecting $\beta_2(t)$ onto $\psi_0(t)$ and $\psi_1(t)$ and then subtracting both these projections from $\beta_2(t)$. For this step, it is *essential* that the basis elements $\{\psi_0(t), \psi_1(t)\}$ be orthonormal. Make sure you understand why. Continuing the derivation, we obtain

$$\begin{aligned}
\alpha_2(t) &= \beta_2(t) - \langle \beta_2(t), \psi_0(t) \rangle \psi_0(t) - \langle \beta_2(t), \psi_1(t) \rangle \psi_1(t) \\
&= \beta_2(t) - \left(\int \beta_2(t)\psi_0(t) \, dt \right) \psi_0(t) - \left(\int \beta_2(t)\psi_1(t) \, dt \right) \psi_1(t) \\
&= \beta_2(t) - 0 - \alpha_1(t) \\
&= \beta_2(t) + \beta_0(t) - \beta_1(t),
\end{aligned}$$

and from this, we find the third basis element as

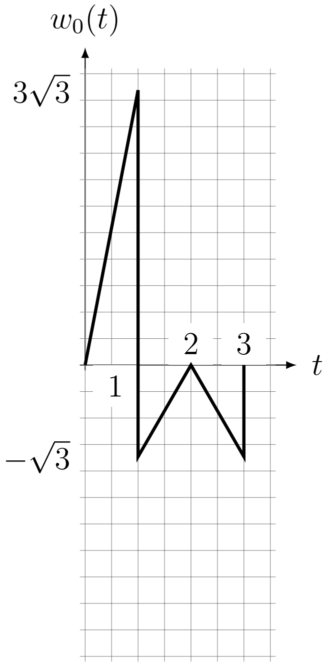
$$\psi_2(t) = \frac{\alpha_2(t)}{||\alpha_2(t)||} = \begin{cases} 0, & \text{if } t < 2 \\ -\sqrt{3}(t-2), & \text{if } 2 \le t \le 3 \\ 0, & \text{if } t > 3 \end{cases} .$$

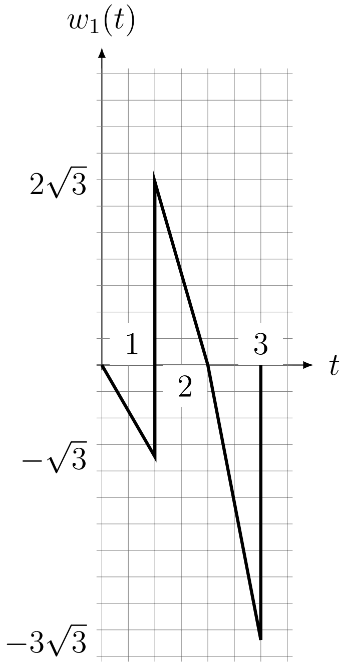
(b) By definition we can write $w_0(t)$ and $w_1(t)$ as follows

$$w_0(t) = 3\psi_0(t) - \psi_1(t) + \psi_2(t) = \begin{cases} 3\sqrt{3}t, & \text{if } 0 \le t < 1 \\ \sqrt{3}(t-2), & \text{if } 1 < t < 2 \\ -\sqrt{3}(t-2), & \text{if } 2 < t \le 3 \end{cases}$$

and

$$w_1(t) = -\psi_0(t) + 2\psi_1(t) + 3\psi_2(t) = \begin{cases} -\sqrt{3}t, & \text{if } 0 \le t < 1 \\ -2\sqrt{3}(t-2), & \text{if } 1 < t < 2 \\ -3\sqrt{3}(t-2), & \text{if } 2 < t \le 3 \end{cases} .$$





(c)

$$\langle c_0, c_1 \rangle = -3 \cdot 1 - 1 \cdot 2 + 1 \cdot 3 = -2.$$

We know that $w_0(t)$ and $w_1(t)$ are both real, thus

$$\langle w_0(t), w_1(t) \rangle = \int w_0(t)w_1(t)\ dt = \int_0^1 -9t^2\ dt + \int_1^2 -6(t-2)^2\ dt + \int_2^3 9(t-2)^2\ dt$$
$$= -\int_1^2 6(t-2)^2\ dt = -2.$$

We see that the inner products are equal as expected.

(d)

$$\|c_0\| = \sqrt{\langle c_0, c_0 \rangle} = \sqrt{11},$$
$$\|w_0\|^2 = \int |w_0(t)|^2 \, dt = \int_0^1 27t^2 \, dt + \int_1^3 3(t-2)^2 \, dt = 9 + 2 = 11.$$

We see that the norms are also equal.

Solution 4.

(a)

$$\|g_i\| = \sqrt{T}, \quad i = 1, 2, 3.$$

(b) Z_1 and Z_2 are independent since g_1 and g_2 are orthogonal. Hence Z is a Gaussian random vector $\sim \mathcal{N}(0, \sigma^2 I_2)$, where $\sigma^2 = \frac{N_0}{2}T$.

(c)

$$P_a = \Pr\{Z_1 \in [1,2] \cap Z_2 \in [1,2]\} = \Pr\{Z_1 \in [1,2]\} \Pr\{Z_2 \in [1,2]\}$$
$$= \left[Q\left(\frac{1}{\sigma}\right) - Q\left(\frac{2}{\sigma}\right) \right]^2,$$

where $\sigma^2 = \frac{N_0}{2}T$.

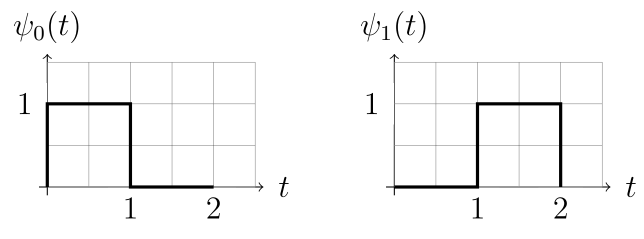
(d) $P_b = P_a$, since one obtains the square (b) from the square (a) via a rotation.

(e) $Z_3 = -Z_1$. $U = Z_1(1,-1)^{\mathsf{T}}$, and thus U can never be in (a), hence $Q_a = 0$.

(f) U is in square (c) if and only if $Z_1 \in [1,2]$. Hence $Q_c = Q\left(\frac{1}{\sigma}\right) - Q\left(\frac{2}{\sigma}\right)$, where $\sigma^2 = \frac{N_0}{2}T$.

Solution 5.

(a) An orthonormal basis for the signal space spanned by the waveforms is[1]:



[1]this can be obtained using Gram-Schmidt procedure or simply by looking at the waveforms

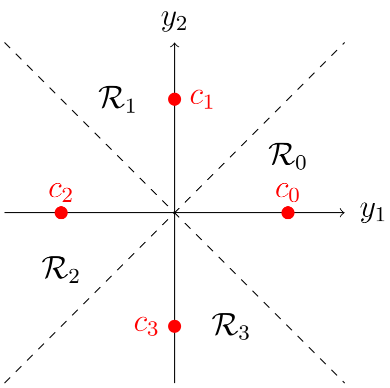
(b) The codewords representing the waveforms are

$$
\begin{aligned}
c_0 &= (\sqrt{\mathcal{E}}, 0) \\
c_1 &= (-\sqrt{\mathcal{E}}, 0) \\
c_2 &= (0, \sqrt{\mathcal{E}}) \\
c_3 &= (0, -\sqrt{\mathcal{E}})
\end{aligned}
$$

(c) As we have seen in the lecture, if $R(t)$ is the noisy received waveform, $(Y_0, Y_1) = (\langle R, \psi_0 \rangle, \langle R, \psi_1 \rangle)$ is a sufficient statistic for decision. Hence, we have the following hypothesis testing problem: Under $H = i$, $i = 0, 1, 2, 3$,

$$Y_i = c_i + Z,$$

where $Z \sim \mathcal{N}(0, \frac{N_0}{2} I_2)$. One can check that c_i, $i = 0, 1, 2, 3$ represent the QPSK codewords (and as we have seen in Homework 3) the decision regions for the ML receiver will be as follows:



The distance between two adjacent codewords (say c_0 and c_1) is $d = \sqrt{2\mathcal{E}}$ and the probability of error of the receiver is

$$
\begin{aligned}
P_e &= 2Q\left(\frac{d}{2\sigma}\right) - Q^2\left(\frac{d}{2\sigma}\right) \\
&= 2Q\left(\frac{\sqrt{2\mathcal{E}}}{2\sqrt{N_0/2}}\right) - Q^2\left(\frac{\sqrt{2\mathcal{E}}}{2\sqrt{N_0/2}}\right) \\
&= 2Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right) - Q^2\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right).
\end{aligned}
$$