SOLUTION 1.

(a) Let the two hypotheses be $H = 0$ and $H = 1$ when $c_0$ and $c_1$ are transmitted, respectively. The ML decision rule is

$$f_{Y_1Y_2|H}(y_1, y_2|1) \overset{H=1}{\gtrless} f_{Y_1Y_2|H}(y_1, y_2|0).$$

Because $Z_1$ and $Z_2$ are independent, we can write

$$\frac{1}{2} e^{-|y_1|-1} \overset{H=1}{\gtrless} \frac{1}{2} e^{-|y_2|-1} \overset{H=0}{\gtrless} \frac{1}{2} e^{-|y_1|+1} \overset{H=1}{\gtrless} \frac{1}{2} e^{-|y_2|+1},$$

and, after taking the logarithm,

$$|y_1 + 1| + |y_2 + 1| \overset{H=1}{\gtrless} |y_1 - 1| + |y_2 - 1| \overset{H=0}{\gtrless}$$

(b) Because the hypotheses are equally likely and $Z_1$ and $Z_2$ have the same distribution, the decision region for $\hat{H} = 0$ contains the points closer to $(-1, -1)$ and the decision region for $\hat{H} = 1$ contains the points closer to $(1, 1)$. For this problem, the distance between the points $(y_{11}, y_{12})$ and $(y_{21}, y_{22})$ is the Manhattan distance, $|y_{11} - y_{21}| + |y_{12} - y_{22}|$, and not the Euclidean distance.

Let us first consider the points above the line $y_2 = -y_1$. It is easy to notice that the points in the positive quadrant are closer to $(1, 1)$ than to $(-1, -1)$, therefore they belong to $R_1 (\hat{H} = 1)$. This is also true if $\{(y_1 \geq 0) \cap (y_2 \in (-1, 0))\}$, or if $\{(y_2 \geq 0) \cap (y_1 \in (-1, 0))\}$. 
Similar reasoning can be applied to the points below the diagonal to determine $\mathcal{R}_0$. The points for which \{(y_1 \leq -1) \cap (y_2 \geq 1)\} or \{(y_1 \geq 1) \cap (y_2 \leq -1)\} are equally distanced to $(-1, -1)$ and $(1, 1)$, therefore they can belong to either $\mathcal{R}_0$ or $\mathcal{R}_1$ with the same probability. This region is named $\mathcal{R}_?$. 

(c) The two hypotheses are equally probable for the region $\mathcal{R}_?$. Therefore, we can split this region in any way between the decision regions and have the same error probability. Because $\mathcal{R}_1$ is included in the region for which $y_2 > -y_1$ and $\mathcal{R}_0$ does not intersect the region for which $y_2 > -y_1$, the error probability is minimized by deciding $\hat{H} = 1$ if $(y_1 + y_2) > 0$.

(d) 

$$P_e(0) = \operatorname{Pr}\{Y_1 + Y_2 > 0|H = 0\}$$

$$= \operatorname{Pr}\{Z_1 + Z_2 - 2 > 0\}$$

$$= \int_2^\infty \frac{e^{-w}}{4}(1 + w) \, dw$$

$$= \left[\frac{-e^{-w}}{4}(w + 2)\right]_2^\infty = e^{-2}.$$ 

By symmetry, and considering that the messages are equally likely, $P_e(0) = P_e(1) = P_e$. 

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Solution 2. We start by normalizing $\beta_1$:

$$
\|\beta_1\| = \sqrt{\langle \beta_1, \beta_1 \rangle} = \sqrt{3}
$$

$$
\psi_1 = \frac{\beta_1}{\|\beta_1\|} = \left( \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right).
$$

We get the next basis vectors as follows:

$$
\langle \psi_1, \beta_2 \rangle = \sqrt{3}
$$

$$
\phi_2 = \beta_2 - \sqrt{3} \psi_1 = (1, 1, -1, 0)
$$

$$
\|\phi_2\| = \sqrt{3}
$$

$$
\psi_2 = \frac{\phi_2}{\|\phi_2\|} = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0 \right).
$$

We compute

$$
\langle \psi_1, \beta_3 \rangle = 0
$$

$$
\langle \psi_2, \beta_3 \rangle = 0.
$$

Thus,

$$
\phi_3 = \beta_3 - 0 \psi_1 - 0 \psi_2 = (1, 0, 1, -2)
$$

$$
\|\phi_3\| = \sqrt{1 + 1 + 4} = \sqrt{6}
$$

$$
\psi_3 = \frac{\phi_3}{\|\phi_3\|} = \left( \frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right).
$$

We proceed similarly to obtain $\phi_4$:

$$
\langle \psi_1, \beta_4 \rangle = \sqrt{3}
$$

$$
\langle \psi_2, \beta_4 \rangle = 0
$$

$$
\langle \psi_3, \beta_4 \rangle = \sqrt{6}
$$

$$
\phi_4 = \beta_4 - \sqrt{3} \psi_1 - 0 \psi_2 - \sqrt{6} \psi_3 = (0, 0, 0, 0).
$$

As can be seen, the last vector is zero. This shows that the dimensionality of the space spanned by $\beta_1, \ldots, \beta_4$ is only 3, not 4. So the other benefit of Gram-Schmidt orthogonalization is that it gives us the dimension of the space spanned by the initial vectors.

Solution 3.

(a) We use the Gram-Schmidt procedure:
(i) The first step is to normalize the function $\beta_0(t)$, i.e. the first function of the basis that we are looking for is

$$\psi_0(t) = \frac{\beta_0(t)}{\|\beta_0(t)\|} = \frac{\beta_0(t)}{\sqrt{\int_0^1 \beta_0(t)^2 \, dt}} = \frac{\beta_0(t)}{\sqrt{\int_0^1 4t^2 \, dt}} = \frac{\sqrt{3}}{2} \beta_0(t) = \begin{cases} 0, & \text{if } t < 0 \\ \sqrt{3}t, & \text{if } 0 \leq t \leq 1 \\ 0, & \text{if } t > 1 \end{cases}.$$

(ii) Next, we subtract from $\beta_1(t)$ the components that are in the span of the currently established part of the basis, i.e. in the span of \{ $\psi_0(t)$ \}. This can be achieved by projecting $\beta_1(t)$ onto $\psi_0(t)$ and then subtracting this projection from $\beta_1(t)$, i.e.

$$\alpha_1(t) = \beta_1(t) - \langle \beta_1(t), \psi_0(t) \rangle \psi_0(t) = \beta_1(t) - \left( \int \beta_1(t) \psi_0(t) \, dt \right) \psi_0(t)$$

$$= \beta_1(t) - \left( \frac{\sqrt{3}}{2} \right) \left( \frac{4}{3} \right) \psi_0(t)$$

$$= \beta_1(t) - 2 \sqrt{3} \psi_0(t)$$

$$= \beta_1(t) - \beta_0(t).$$

From this, we find the second basis element as

$$\psi_1(t) = \frac{\alpha_1(t)}{||\alpha_1(t)||} = \begin{cases} 0, & \text{if } t < 1 \\ -\sqrt{3}(t - 2), & \text{if } 1 \leq t \leq 2 \\ 0, & \text{if } t > 2 \end{cases}.$$

(iii) Again, we subtract from $\beta_2(t)$ the components that are in the span of the currently established part of the basis, i.e. in the span of \{ $\psi_0(t), \psi_1(t)$ \}. This can be achieved by projecting $\beta_2(t)$ onto $\psi_0(t)$ and $\psi_1(t)$ and then subtracting both these projections from $\beta_2(t)$. For this step, it is essential that the basis elements \{ $\psi_0(t), \psi_1(t)$ \} be orthonormal. Make sure you understand why. Continuing the derivation, we obtain

$$\alpha_2(t) = \beta_2(t) - \langle \beta_2(t), \psi_0(t) \rangle \psi_0(t) - \langle \beta_2(t), \psi_1(t) \rangle \psi_1(t)$$

$$= \beta_2(t) - \left( \int \beta_2(t) \psi_0(t) \, dt \right) \psi_0(t) - \left( \int \beta_2(t) \psi_1(t) \, dt \right) \psi_1(t)$$

$$= \beta_2(t) - 0 - \alpha_1(t)$$

$$= \beta_2(t) + \beta_0(t) - \beta_1(t),$$
and from this, we find the third basis element as
\[
\psi_2(t) = \frac{\alpha_2(t)}{||\alpha_2(t)||} = \begin{cases} 
0, & \text{if } t < 2 \\
-\sqrt{3}(t - 2), & \text{if } 2 \leq t \leq 3 \\
0, & \text{if } t > 3
\end{cases}
\]
(b) By definition we can write \(w_0(t)\) and \(w_1(t)\) as follows
\[
w_0(t) = 3\psi_0(t) - \psi_1(t) + \psi_2(t) = \begin{cases} 
3\sqrt{3}t, & \text{if } 0 \leq t < 1 \\
\sqrt{3}(t - 2), & \text{if } 1 < t < 2 \\
-\sqrt{3}(t - 2), & \text{if } 2 < t \leq 3
\end{cases}
\]
and
\[
w_1(t) = -\psi_0(t) + 2\psi_1(t) + 3\psi_2(t) = \begin{cases} 
-\sqrt{3}t, & \text{if } 0 \leq t < 1 \\
-2\sqrt{3}(t - 2), & \text{if } 1 < t < 2 \\
-3\sqrt{3}(t - 2), & \text{if } 2 < t \leq 3
\end{cases}
\]
(c)
\[
\langle c_0, c_1 \rangle = -3 \cdot 1 - 1 \cdot 2 + 1 \cdot 3 = -2.
\]
We know that \(w_0(t)\) and \(w_1(t)\) are both real, thus
\[
\langle w_0(t), w_1(t) \rangle = \int w_0(t)w_1(t) \, dt = \int_1^1 -9t^2 \, dt + \int_1^2 -6(t - 2)^2 \, dt + \int_2^3 9(t - 2)^2 \, dt
\]
\[
= - \int_1^2 6(t - 2)^2 \, dt = -2.
\]
We see that the inner products are equal as expected.
We see that the norms are also equal.

**Solution 4.**

(a) \[ \|g_i\| = \sqrt{T}, \quad i = 1, 2, 3. \]

(b) \( Z_1 \) and \( Z_2 \) are independent since \( g_1 \) and \( g_2 \) are orthogonal. Hence \( Z \) is a Gaussian random vector \( \sim \mathcal{N}(0, \sigma^2 I_2) \), where \( \sigma^2 = \frac{N_0}{2} T \).

(c) \[ P_a = \Pr \{ Z_1 \in [1,2] \cap Z_2 \in [1,2] \} = \Pr \{ Z_1 \in [1,2] \} \Pr \{ Z_2 \in [1,2] \} = \left[ Q\left(\frac{1}{\sigma}\right) - Q\left(\frac{2}{\sigma}\right)\right]^2, \]
where \( \sigma^2 = \frac{N_0}{2} T \).

(d) \( P_b = P_a \), since one obtains the square (b) from the square (a) via a rotation.

(e) \( Z_3 = -Z_1 \). \( U = Z_1(1, -1)^T \), and thus \( U \) can never be in (a), hence \( Q_a = 0 \).

(f) \( U \) is in square (c) if and only if \( Z_1 \in [1,2] \). Hence \( Q_c = Q\left(\frac{1}{\sigma}\right) - Q\left(\frac{2}{\sigma}\right) \), where \( \sigma^2 = \frac{N_0}{2} T \).

**Solution 5.**

(a) An orthonormal basis for the signal space spanned by the waveforms is\(^1\):

\[ \psi_0(t) \quad \psi_1(t) \]

\[ \begin{array}{c}
1 \\
1 \\
1 \\
\end{array} \]
\[ \begin{array}{c}
1 \\
1 \\
1 \\
\end{array} \]

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\( ^1 \)This can be obtained using Gram-Schmidt procedure or simply by looking at the waveforms.
(b) The codewords representing the waveforms are
\[ c_0 = (\sqrt{E}, 0) \]
\[ c_1 = (-\sqrt{E}, 0) \]
\[ c_2 = (0, \sqrt{E}) \]
\[ c_3 = (0, -\sqrt{E}) \]

(c) As we have seen in the lecture, if \( R(t) \) is the noisy received waveform, \( (Y_0, Y_1) = (\langle R, \psi_0 \rangle, \langle R, \psi_1 \rangle) \) is a sufficient statistic for decision. Hence, we have the following hypothesis testing problem: Under \( H = i, \ i = 0, 1, 2, 3 \),
\[ Y_i = c_i + Z, \]
where \( Z \sim \mathcal{N}(0, \frac{N_0}{2} I_2) \). One can check that \( c_i, \ i = 0, 1, 2, 3 \) represent the QPSK codewords (and as we have seen in Homework 3) the decision regions for the ML receiver will be as follows:

The distance between two adjacent codewords (say \( c_0 \) and \( c_1 \)) is \( d = \sqrt{2E} \) and the probability of error of the receiver is
\[
P_e = 2Q \left( \frac{d}{2\sigma} \right) - Q^2 \left( \frac{d}{2\sigma} \right) = 2Q \left( \frac{\sqrt{2E}}{2\sqrt{N_0}/2} \right) - Q^2 \left( \frac{\sqrt{2E}}{2\sqrt{N_0}/2} \right) = 2Q \left( \frac{\sqrt{E}}{\sqrt{N_0}} \right) - Q^2 \left( \frac{\sqrt{E}}{\sqrt{N_0}} \right).
\]