ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

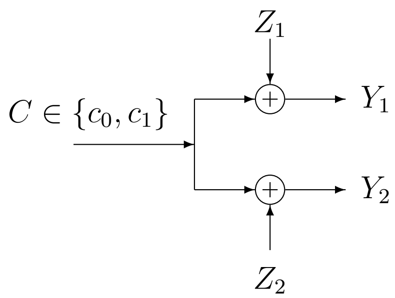
School of Computer and Communication Sciences

Handout 10 — Principles of Digital Communications

Problem Set 5 — Mar. 18, 2015

PROBLEM 1. *(SIMO Channel with Laplacian Noise)* One of the two signals $c_0 = -1, c_1 = 1$ is transmitted over the channel shown in the diagram below. The two noise random variables Z_1 and Z_2 are statistically independent of the transmitted signal and of each other. Their density functions are

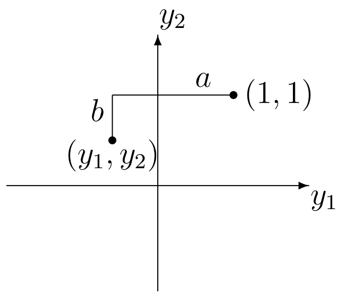
$$f_{Z_1}(\alpha) = f_{Z_2}(\alpha) = \frac{1}{2}\, e^{-|\alpha|}.$$



(a) Derive a maximum likelihood decision rule.

(b) Describe the maximum likelihood decision regions in the (y_1, y_2) plane. Try to describe the "Either Choice" regions, i.e., the regions in which it does not matter if you decide for c_0 or for c_1.

Hint: Use geometric reasoning and the fact that for a point (y_1, y_2) as shown below, $|y_1 - 1| + |y_2 - 1| = a + b$.



(c) A receiver decides that c_1 was transmitted if and only if $(y_1 + y_2) > 0$. Does this receiver minimize the error probability for equally likely messages?

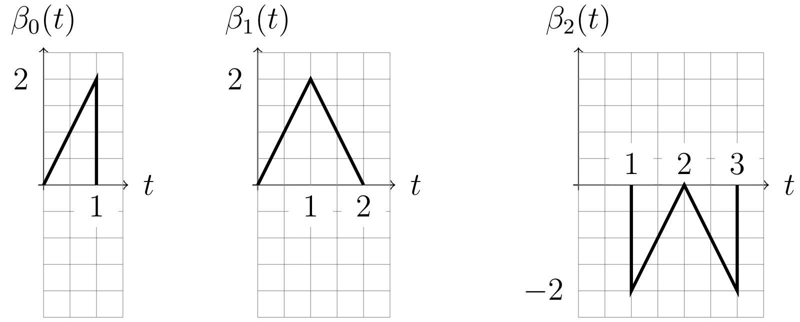
(d) What is the error probability for the receiver in (c)?

Hint: One way to do this is to use the fact that if $W = Z_1 + Z_2$ then $f_W(w) = \frac{e^{-w}}{4}(1+w)$ for $w > 0$.

PROBLEM 2. *(Gram-Schmidt Procedure On Tuples)* Use the Gram-Schmidt orthogonalization procedure to find an orthonormal basis for the subspace spanned by the vectors $\beta_1, \ldots, \beta_4$ where $\beta_1 = (1,0,1,1)^{\mathsf{T}}$, $\beta_2 = (2,1,0,1)^{\mathsf{T}}$, $\beta_3 = (1,0,1,-2)^{\mathsf{T}}$, and $\beta_4 = (2,0,2,-1)^{\mathsf{T}}$.

PROBLEM 3. *(Gram-Schmidt Procedure on Three Waveforms)*

(a) By means of the Gram-Schmidt procedure, find an orthonormal basis for the signal space spanned by the waveforms in the figure below.

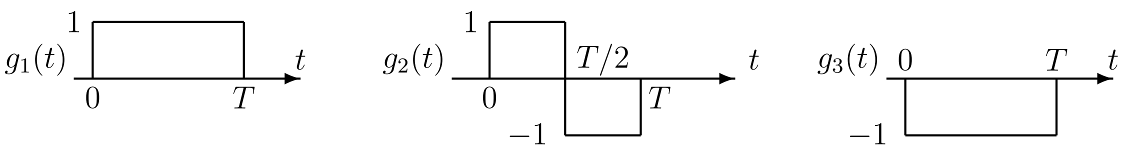


(b) In your chosen orthonormal basis, let $w_0(t)$ and $w_1(t)$ be represented by the codewords $c_0 = (3,-1,1)^{\mathsf{T}}$ and $c_1 = (-1,2,3)^{\mathsf{T}}$, respectively. Plot $w_0(t)$ and $w_1(t)$.

(c) Compute the inner products $\langle c_0, c_1\rangle$ and $\langle w_0, w_1\rangle$ and compare them.

(d) Compute the norms $\|c_0\|$ and $\|w_0\|$ and compare them.

PROBLEM 4. *(Noise in Regions)* Let $N(t)$ be a white Gaussian noise of power spectral density $\frac{N_0}{2}$. Let $g_1(t)$, $g_2(t)$, and $g_3(t)$ be waveforms as shown in the following figure. For $i = 1,2,3$, let $Z_i = \int N(t) g_i^*(t)\, dt$. Then, define $Z = (Z_1, Z_2)^{\mathsf{T}}$ and $U = (Z_1, Z_3)^{\mathsf{T}}$.



(a) Determine the norm $\|g_i\|$, $i = 1,2,3$.

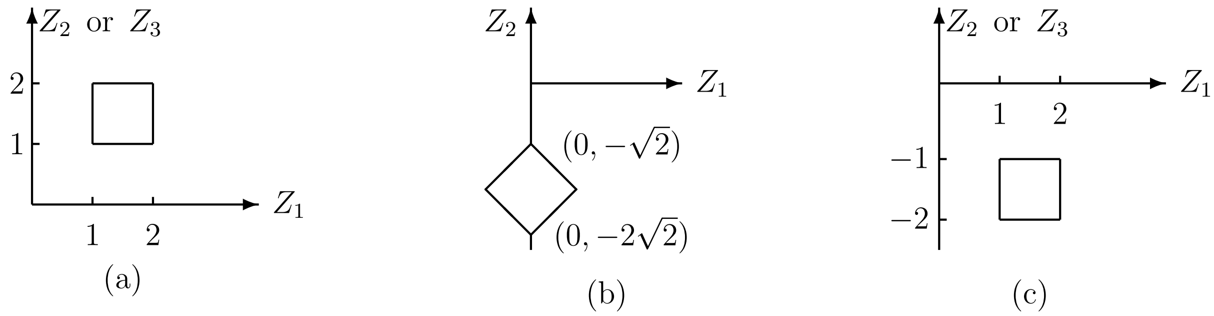
(b) Are Z_1 and Z_2 independent? Justify your answer.

(c) Find the probability P_a that Z lies in the square labeled (a) in the figure below.

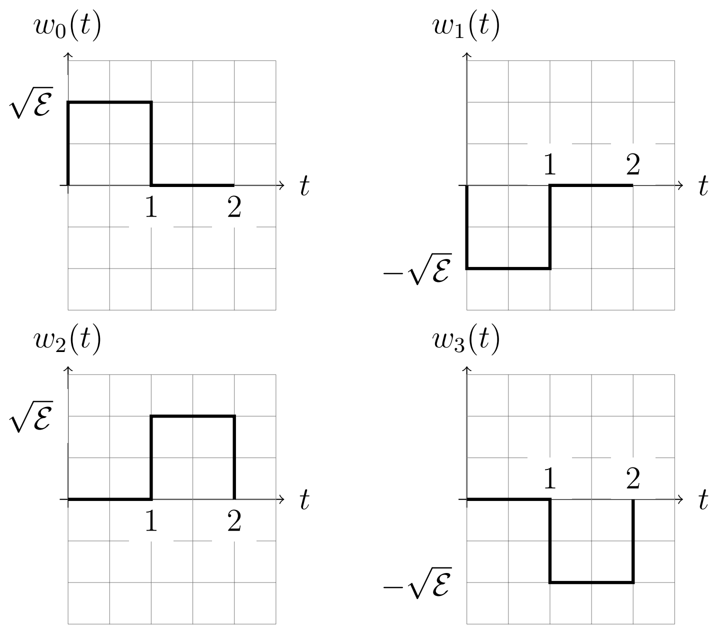
(d) Find the probability P_b that Z lies in the square (b) of the same figure.

(e) Find the probability Q_a that U lies in the square (a).

(f) Find the probability Q_c that U lies in the square (c).



PROBLEM 5. *(4-PSK Signaling)* Consider the four waveforms $w_k(t)$, $k = 0, 1, 2, 3$ represented in the figure below.



(a) Determine an orthonormal basis for the signal space spanned by these waveforms.

Hint: No lengthy calculations needed.

(b) Determine the codewords c_i, $i = 0, 1, 2, 3$ representing the waveforms.

(c) Assume a transmitter sends w_i to communicate a digit $i \in \{0, 1, 2, 3\}$ across a continuous-time AWGN channel of power spectral density $\frac{N_0}{2}$. Write an expression for the error probability of the ML receiver in terms of $\mathcal{E}$ and N_0.