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Handout 9 Principles of Digital Communications

Solutions to Problem Set 4 Mar. 17, 2015

SOLUTION 1.

(a) The MAP decision rule can always be written as

$$\begin{aligned}
\hat{H}(y) &= \arg\max_i f_{Y|H}(y|i)P_H(i) \\
&= \arg\max_i g_i(T(y))h(y)P_H(i) \\
&= \arg\max_i g_i(T(y))P_H(i).
\end{aligned}$$

The last step is valid because $h(y)$ is a non-negative constant which is independent of i and thus does not give any further information for our decision.

(b) Let us define the event $\mathcal{B} = \{y : T(y) = t\}$. Then,

$$f_{Y|H,T(Y)}(y|i,t) = \frac{f_{Y|H}(y|i)\mathbb{1}_{\mathcal{B}}(y)}{\int_{\mathcal{B}} f_{Y|H}(y|i)dy}.$$

If $f_{Y|H}(y|i) = g_i(T(y))h(y)$, then

$$\begin{aligned}
f_{Y|H,T(Y)}(y|i,t) &= \frac{g_i(T(y))h(y)\mathbb{1}_{\mathcal{B}}(y)}{\int_{\mathcal{B}} g_i(T(y))h(y)dy} \\
&= \frac{g_i(t)h(y)\mathbb{1}_{\mathcal{B}}(y)}{g_i(t)\int_{\mathcal{B}} h(y)dy} \\
&= \frac{h(y)\mathbb{1}_{\mathcal{B}}(y)}{\int_{\mathcal{B}} h(y)dy}.
\end{aligned}$$

Hence, we see that $f_{Y|H,T(Y)}(y|i,t)$ does not depend on i, so $H \to T(Y) \to Y$.

SOLUTION 2.

(a) Since Y is an i.i.d. sequence,

$$\begin{aligned}
P_{Y|H}(y|i) = \prod_{k=1}^{n} P_{Y_k|H}(y_k|i) &= \frac{\lambda_i^{\sum_{k=1}^n y_k}}{\prod_{k=1}^n (y_k)!} e^{-n\lambda_i} \\
&= \underbrace{e^{-n\lambda_i}\lambda_i^{n\left(\frac{1}{n}\sum_{k=1}^n y_k\right)}}_{g_i(T(y))} \underbrace{\frac{1}{\prod_{k=1}^n (y_k)!}}_{h(y)}.
\end{aligned}$$

(b) Since $Z_1, \ldots, Z_n$ are i.i.d. additive noise samples,

$$f_{Y|H}(y|i) = \prod_{k=1}^{n} f_Z(y_k - \theta_i) = \lambda^n e^{-\lambda \sum_{k=1}^{n}(y_k - \theta_i)} \mathbb{1}\{y_k \geq \theta_i : \forall k = 1, \ldots, n\}$$
$$= \underbrace{\lambda^n e^{-\lambda n\left(\frac{1}{n}\sum_{k=1}^{n} y_k - \theta_i\right)} \mathbb{1}\left\{\min_{k=1,\ldots,n} y_k \geq \theta_i\right\}}_{g_i(T(y))}$$

with $h(y) = 1$.

Solution 3. If $H = 0$, we have $Y_2 = Z_1 Z_2 = Y_1 Z_2$, and if $H = 1$, we have $Y_2 = -Z_1 Z_2 = Y_1 Z_2$. Therefore, $Y_2 = Y_1 Z_2$ in all cases. Now since Z_2 is independent of H, we clearly have $H \to Y_1 \to (Y_1, Z_2 Y_1)$. Hence, Y_1 is a sufficient statistic.

Solution 4.

(a) The MAP decoder $\hat{H}(y)$ is given by

$$\hat{H}(y) = \arg\max_i P_{Y|H}(y|i) = \begin{cases} 0 & \text{if } y = 0 \text{ or } y = 1 \\ 1 & \text{if } y = 2 \text{ or } y = 3. \end{cases}$$

$T(Y)$ takes two values with the conditional probabilities

$$P_{T|H}(t|0) = \begin{cases} 0.7 & \text{if } t = 0 \\ 0.3 & \text{if } t = 1 \end{cases} \qquad P_{T|H}(t|1) = \begin{cases} 0.3 & \text{if } t = 0 \\ 0.7 & \text{if } t = 1. \end{cases}$$

Therefore, the MAP decoder $\hat{H}(T(y))$ is

$$\hat{H}(T(y)) = \arg\max_i P_{T(Y)|H}(t|i) = \begin{cases} 0 & \text{if } t = 0 \quad (y = 0 \text{ or } y = 1) \\ 1 & \text{if } t = 1 \quad (y = 2 \text{ or } y = 3). \end{cases}$$

Hence, the two decoders are equivalent.

(b) We have

$$\Pr\{Y = 0|T(Y) = 0, H = 0\} = \frac{\Pr\{Y = 0, T(Y) = 0|H = 0\}}{\Pr\{T(Y) = 0|H = 0\}} = \frac{0.4}{0.7} = \frac{4}{7}$$

and

$$\Pr\{Y = 0|T(Y) = 0, H = 1\} = \frac{\Pr\{Y = 0, T(Y) = 0|H = 1\}}{\Pr\{T(Y) = 0|H = 1\}} = \frac{0.1}{0.3} = \frac{1}{3}.$$

Thus $\Pr\{Y = 0|T(Y) = 0, H = 0\} \neq \Pr\{Y = 0|T(Y) = 0, H = 1\}$, hence $H \to T(Y) \to Y$ is not true, although the MAP decoders are equivalent.

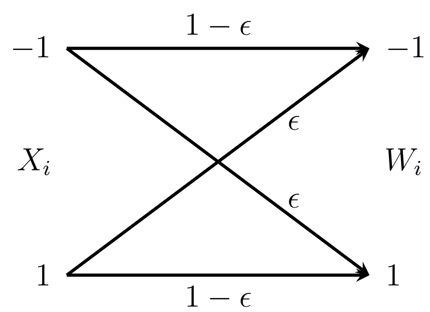
Solution 5.

(a) Let $X_i = c_{H,i}$ be the i-th symbol that was sent, i.e., $X_i = 1$ if $H = 0$ and $X_i = -1$ if $H = 1$. We have:

$$P_{W_i|X_i}(1|-1) = \Pr\{Y_i > 0|H = 1\} = \Pr\{-1 + Z > 0\} = Q\left(\frac{1}{\sigma}\right).$$

Similarly, we can show that $P_{W_i|X_i}(-1|-1) = 1 - Q\left(\frac{1}{\sigma}\right)$, $P_{W_i|X_i}(-1|1) = Q\left(\frac{1}{\sigma}\right)$ and $P_{W_i|X_i}(1|1) = 1 - Q\left(\frac{1}{\sigma}\right)$.

The overall system between X_i and W_i may be viewed as a channel with input 1 or -1 and output also 1 or -1. There is a certain probability ϵ (called *transition* or *crossover* probability, and which is equal to $Q\left(\frac{1}{\sigma}\right)$ in our case) that the channel converts 1 into -1 or vice versa:



This particular channel is called the *Binary Symmetric Channel.* Various results can be found easily from the above diagram. For instance, it is clear that if we put n consecutive 1's into the channel, the probability of getting, at the output, a particular sequence $(w_1, \ldots, w_n)$ which contains exactly k 1's is simply $(1-\epsilon)^k \epsilon^{n-k}$. Similarly, the probability of getting, at the output, *any* sequence that contains exactly k 1's is $\binom{n}{k}(1-\epsilon)^k\epsilon^{n-k}$ because there are $\binom{n}{k}$ distinct sequences with exactly k ones each, and every one of them has probability $(1-\epsilon)^k\epsilon^{n-k}$.

The MAP decision rule is

$$\frac{P_{W_1\ldots W_n|H}(w_1, \ldots, w_n|1)}{P_{W_1\ldots W_n|H}(w_1, \ldots, w_n|0)} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{P_H(0)}{P_H(1)} = 1 \quad \text{or,}$$

$$\frac{\epsilon^k(1-\epsilon)^{n-k}}{(1-\epsilon)^k\epsilon^{n-k}} = \left(\frac{\epsilon}{1-\epsilon}\right)^{2k-n} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 1.$$

The expression only depends on k, therefore the number of ones in the received sequence is a sufficient statistic.

Taking the logarithm, we obtain

$$(2k-n)\log\left(\frac{\epsilon}{1-\epsilon}\right) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 0.$$

Since $\epsilon < 1/2$, $\log\left(\frac{\epsilon}{1-\epsilon}\right) < 0$, and thus, when we divide by this term, the direction of the inequality is changed. Using this, the decision rule can be written as

$$k \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} \frac{n}{2}.$$

That is, the best decision rule is simply *majority voting*: if the majority of the received values is 1, we decide for hypothesis $H = 0$ (i.e. the transmitted value was 1). If the majority of the received values is -1, we decide for hypothesis $H = 1$ (i.e. the transmitted value was -1).

(b) Let us assume that n is odd. Then,

$$\begin{aligned} P_e(0) &= \Pr\{k < n/2 | H = 0\} \\ &= \sum_{m=0}^{(n-1)/2} \binom{n}{m} (1-\epsilon)^m \epsilon^{n-m}. \end{aligned}$$

By the symmetry of the problem, $P_e(1)$ has the same value. Thus,

$$\tilde{P}_e = \sum_{m=0}^{(n-1)/2} \binom{n}{m} (1-\epsilon)^m \epsilon^{n-m}.$$

If n is even, we introduce a slight asymmetry because the term for $n/2$ has to be assigned to either $H = 0$ or $H = 1$.

Because this sum cannot be evaluated explicitly, in the following, we bound it using the *Bhattacharyya bound*.

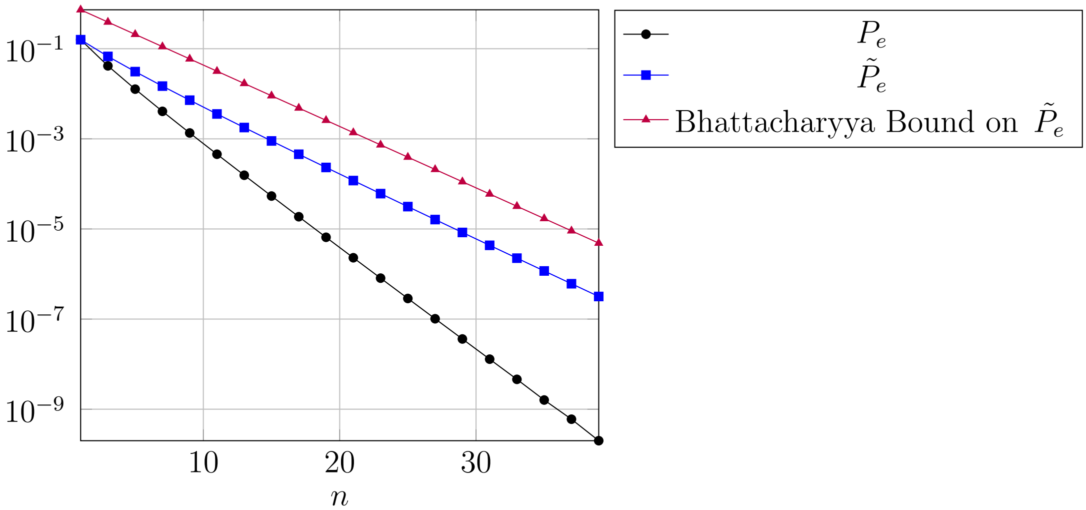
(c) The general formula for the Bhattacharyya bound is

$$\tilde{P}_e \leq \sum_i \sum_{j:j \neq i} \sqrt{P_H(i) P_H(j)} \int_{w \in \mathbb{R}^n} \sqrt{f_{W|H}(w|i) f_{W|H}(w|j)}\, dw.$$

In our case, this becomes

$$\begin{aligned} \tilde{P}_e &\leq 2 \frac{1}{2} \sum_w \sqrt{P_{W|H}(w|0) P_{W|H}(w|1)} \\ &= \sum_w \sqrt{(1-\epsilon)^{k(w)} \epsilon^{n-k(w)}\ \epsilon^{k(w)} (1-\epsilon)^{n-k(w)}} \\ &= \sum_w \sqrt{\epsilon^n (1-\epsilon)^n} \\ &= 2^n \sqrt{\epsilon^n (1-\epsilon)^n}. \end{aligned}$$

(d) Again, we assume that n is odd. The following plot shows the error probabilities for various values of n:



SOLUTION 6.

(a) Inequality (a) follows from the *Bhattacharyya Bound*.

Using the definition of DMC, it is straightforward to see that

$$P_{Y|X}(y|c_0) = \prod_{i=1}^{n} P_{Y|X}(y_i|c_{0,i}) \qquad \text{and}$$
$$P_{Y|X}(y|c_1) = \prod_{i=1}^{n} P_{Y|X}(y_i|c_{1,i}).$$

(b) follows by substituting the above values in (a).

Equality (c) is obtained by observing that $\sum_y$ is the same as $\sum_{y_1,\ldots,y_n}$ (the first one being a vector notation for the sum over all possible $y_1, \ldots, y_n$).

In (c), we see that we want the sum of all possible products. This is the same as summing over each y_i and taking the product of the resulting sum for all y_i. This results in equality (d). We obtain (e) by writing (d) in a more concise form.

When $c_{0,i} = c_{1,i}$, $\sqrt{P_{Y|X}(y|c_{0,i})P_{Y|X}(y|c_{1,i})} = P_{Y|X}(y|c_{0,i})$. Therefore,

$$\sum_y \sqrt{P_{Y|X}(y|c_{0,i})P_{Y|X}(y|c_{1,i})} = \sum_y P_{Y|X}(y|c_{0,i}) = 1.$$

This does not affect the product, so we are only interested in the terms where $c_{0,i} \neq c_{1,i}$. We form the product of all such sums where $c_{0,i} \neq c_{1,i}$. We then look out for terms where $c_{0,i} = a$ and $c_{1,i} = b, a \neq b$, and raise the sum to the appropriate power. (e.g. If we have the product $prpqrpqrr$, we would write it as $p^3q^2r^4$). Hence equality (f).

(b) For a binary input channel, we have only two source symbols $\mathcal{X} = \{a, b\}$. Thus,

$$\begin{aligned} P_e &\leq z^{n(a,b)} z^{n(b,a)} \\ &= z^{n(a,b)+n(b,a)} \\ &= z^{d_H(c_0,c_1)}. \end{aligned}$$

(c) The value of z is:

(i) For a binary input Gaussian channel,

$$\begin{aligned} z &= \int_y \sqrt{f_{Y|X}(y|0) f_{Y|X}(y|1)}\ dy \\ &= \exp\left(-\frac{E}{2\sigma^2}\right). \end{aligned}$$

(ii) For the Binary Symmetric Channel (BSC),

$$\begin{aligned} z &= \sqrt{\Pr\{y=0|x=0\} \Pr\{y=0|x=1\}} + \sqrt{\Pr\{y=1|x=0\} \Pr\{y=1|x=1\}} \\ &= 2\sqrt{\delta(1-\delta)}. \end{aligned}$$

(iii) For the Binary Erasure Channel (BEC),

$$\begin{aligned} z &= \sqrt{\Pr\{y=0|x=0\} \Pr\{y=0|x=1\}} + \sqrt{\Pr\{y=E|x=0\} \Pr\{y=E|x=1\}} \\ &\quad + \sqrt{\Pr\{y=1|x=0\} \Pr\{y=1|x=1\}} \\ &= 0 + \delta + 0 \\ &= \delta. \end{aligned}$$

Solution 7.

(a) From the definition of the decision region $\mathcal{R}_i$,

$$\mathcal{R}_i = \left\{ y : P_H(i) f_{Y|H}(y|i) \geq P_H(j) f_{Y|H}(y|j) \right\} \quad i \neq j,$$

it is easy to see that in region $\mathcal{R}_0$

$$P_H(0) f_{Y|H}(y|0) \geq P_H(1) f_{Y|H}(y|1)$$

and vice-versa. Thus we can write

$$\begin{aligned} P_e &= P_H(0) \int_{\mathcal{R}_1} f_{Y|H}(y|0)\ dy + P_H(1) \int_{\mathcal{R}_0} f_{Y|H}(y|1)\ dy \\ &= \int_{\mathcal{R}_1} \min\{P_H(0) f_{Y|H}(y|0), P_H(1) f_{Y|H}(y|1)\}\ dy \\ &\quad + \int_{\mathcal{R}_0} \min\{P_H(0) f_{Y|H}(y|0), P_H(1) f_{Y|H}(y|1)\}\ dy \\ &= \int_{\mathcal{R}_0 \cup \mathcal{R}_1} \min\{P_H(0) f_{Y|H}(y|0), P_H(1) f_{Y|H}(y|1)\}\ dy \\ &= \int_y \min\{P_H(0) f_{Y|H}(y|0), P_H(1) f_{Y|H}(y|1)\}\ dy. \end{aligned}$$

(b) Without loss of generality, let us assume that $a \leq b$. Then $\sqrt{b/a} \geq 1$ and $\min(a,b) = a \leq a\sqrt{b/a} = \sqrt{ab}$.

To show that for $a, b \geq 0, \sqrt{ab} \leq \frac{a+b}{2}$, we proceed as follows. Let $m = (a+b)/2$ be the midpoint of an imaginary segment of the real line that goes from a to b. Let $d = (b-a)/2$ be half the distance between a and b. Writing a and b in terms of m and d we obtain $ab = (m-d)(m+d) = m^2 - d^2 \leq m^2$, which is the desired result.

Considering this, we can write

$$\begin{aligned}
P_e &= \int_y \min\left\{P_H(0) f_{Y|H}(y|0), P_H(1) f_{Y|H}(y|1)\right\} \, dy \\
&\leq \int_y \sqrt{P_H(0) f_{Y|H}(y|0) P_H(1) f_{Y|H}(y|1)} \, dy \\
&= \sqrt{P_H(0) P_H(1)} \int_y \sqrt{f_{Y|H}(y|0) f_{Y|H}(y|1)} \, dy \\
&\leq \frac{P_H(0) + P_H(1)}{2} \int_y \sqrt{f_{Y|H}(y|0) f_{Y|H}(y|1)} \, dy \\
&= \frac{1}{2} \int_y \sqrt{f_{Y|H}(y|0) f_{Y|H}(y|1)} \, dy.
\end{aligned}$$

(c) In the book, we upper-bound $P_e(i)$ individually instead of upper-bounding the final result, $P_e = \sum_i P_H(i) P_e(i)$. For the binary case, this is equivalent to

$$\begin{aligned}
P_e(0) &= \int_{\mathcal{R}_1} f_{Y|H}(y|0) \, dy \\
&= \int_{\mathcal{R}_1} \min\left\{f_{Y|H}(y|0), f_{Y|H}(y|1)\right\} \, dy \\
&\leq \int_{\mathcal{R}_1} \sqrt{f_{Y|H}(y|0) f_{Y|H}(y|1)} \, dy \\
&\leq \int_y \sqrt{f_{Y|H}(y|0) f_{Y|H}(y|1)} \, dy.
\end{aligned}$$

The last step, which further loosens the bound, is necessary to find a bound of $P_e(0)$ that does not depend on $\mathcal{R}_1$. This "over-bounding" is avoided in (b) by finding the bound over the whole P_e.