ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 6 Principles of Digital Communications

Problem Set 3 Mar. 4, 2015

PROBLEM 1. Consider the ternary hypothesis testing problem

$$H_0 : Y = c_0 + Z, \qquad H_1 : Y = c_1 + Z, \qquad H_2 : Y = c_2 + Z,$$

where $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ is the two-dimensional observation vector,

$$c_0 = \sqrt{\mathcal{E}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad c_1 = \frac{1}{2}\sqrt{\mathcal{E}} \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}, \qquad c_2 = \frac{1}{2}\sqrt{\mathcal{E}} \begin{bmatrix} -1 \\ -\sqrt{3} \end{bmatrix},$$

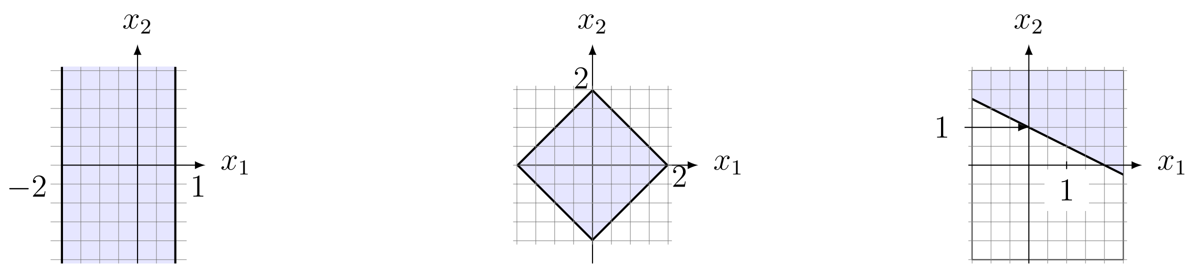
and $Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \sim \mathcal{N}(0, \sigma^2 I_2)$.

(a) Assuming the three hypotheses are equally likely, draw the optimal decision regions in the (Y_1, Y_2) plane.

(b) Assume now that the a-priori probabilities for the hypotheses are given by $\Pr\{H = 0\} = \frac{1}{2}$, $\Pr\{H = 1\} = \Pr\{H = 2\} = \frac{1}{4}$. Draw the decision regions in the (L_1, L_2) plane where

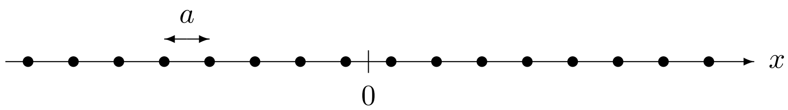
$$L_i := \frac{f_{Y|H}(Y|i)}{f_{Y|H}(Y|0)}, \quad i = 1, 2.$$

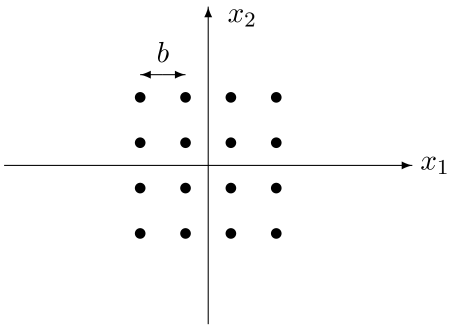
PROBLEM 2. *(Q-Function on Regions)* Let $X \sim \mathcal{N}(0, \sigma^2 I_2)$. For each of the three figures below, express the probability that X lies in the shaded region. You may use the Q-function when appropriate.



(a) (b) (c)

PROBLEM 3. *(16-PAM versus 16-QAM)* The two signal constellations depicted below are used to communicate across an additive white Gaussian noise channel. Let the noise variance be σ^2. Each point represents a codeword c_i for some i. Assume each codeword is used with the same probability.





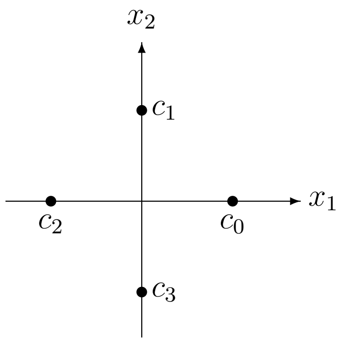
(a) For each signal constellation, compute the average probability of error P_e as a function of the parameters a and b, respectively.

(b) For each signal constellation, compute the average energy per symbol $\mathcal{E}$ as a function of the parameters a and b, respectively:

$$\mathcal{E} = \sum_{i=1}^{16} P_H(i) \|c_i\|^2.$$

(c) Plot P_e versus $\mathcal{E}/\sigma^2$ for both signal constellations and comment.

Problem 4. *(QPSK Decision Regions)* Let $H \in \{0,1,2,3\}$ and assume that when $H = i$ you transmit the codeword c_i shown below. Under $H = i$, the receiver observes $Y = c_i + Z$.

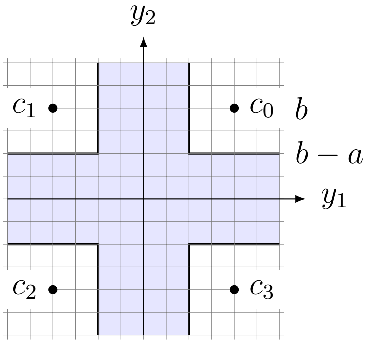


(a) Draw the decoding regions assuming that $Z \sim \mathcal{N}(0, \sigma^2 I_2)$ and that $P_H(i) = 1/4$, $i \in \{0,1,2,3\}$.

(b) Draw the decoding regions (qualitatively) assuming $Z \sim \mathcal{N}(0, \sigma^2 I_2)$ and $P_H(0) = P_H(2) > P_H(1) = P_H(3)$. Justify your answer.

(c) Assume again that $P_H(i) = 1/4,\ i \in \{0,1,2,3\}$ and that $Z \sim \mathcal{N}(0,K)$, where $K = \begin{pmatrix} \sigma^2 & 0 \\ 0 & 4\sigma^2 \end{pmatrix}$. How do you decode now?

Problem 5. *(QAM with Erasure)* Consider a QAM receiver that outputs a special symbol δ (called erasure) whenever the observation falls in the shaded area shown in the figure below. Assume that $c_0 \in \mathbb{R}^2$ is transmitted and that $Y = c_0 + N$ is received where $N \sim \mathcal{N}(0, \sigma^2 I_2)$. Let P_{0i}, $i = 0,1,2,3$ be the probability that the receiver outputs $\hat{H} = i$ and let $P_{0\delta}$ be the probability that it outputs δ. Determine P_{00}, P_{01}, P_{02}, P_{03} and $P_{0\delta}$.



Remark. *If we choose $b-a$ large enough, we can make sure that the probability of the error is very small (we say that an error occurred if $\hat{H} = i$, $i \in \{0,1,2,3\}$ and $H \neq \hat{H}$). When $\hat{H} = \delta$, the receiver can ask for a retransmission of H. This requires a feedback channel from the receiver to the sender. In most practical applications, such a feedback channel is available.*

Problem 6. *(Antenna Array)* The following problem relates to the design of multi-antenna systems. Consider the binary equiprobable hypothesis testing problem:

$$\begin{aligned} H = 0 &: Y_1 = A + Z_1, \quad Y_2 = A + Z_2 \\ H = 1 &: Y_1 = -A + Z_1, \quad Y_2 = -A + Z_2, \end{aligned}$$

where Z_1, Z_2 are independent Gaussian random variables with *different* variances $\sigma_1^2 \neq \sigma_2^2$, that is, $Z_1 \sim \mathcal{N}(0, \sigma_1^2)$ and $Z_2 \sim \mathcal{N}(0, \sigma_2^2)$. $A > 0$ is a constant.

(a) Show that the decision rule that minimizes the probability of error (based on the observable Y_1 and Y_2) can be stated as

$$\sigma_2^2 y_1 + \sigma_1^2 y_2 \ \underset{1}{\overset{0}{\gtrless}}\ 0.$$

(b) Draw the decision regions in the (Y_1, Y_2) plane for the special case where $\sigma_1 = 2\sigma_2$.

(c) Evaluate the probability of the error for the optimal detector as a function of σ_1^2, σ_2^2 and A.