PROBLEM 1. Consider the ternary hypothesis testing problem

\[ H_0 : Y = c_0 + Z, \quad H_1 : Y = c_1 + Z, \quad H_2 : Y = c_2 + Z, \]

where \( Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \) is the two-dimensional observation vector,

\[
c_0 = \sqrt{\mathbf{E}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c_1 = \frac{1}{2} \sqrt{\mathbf{E}} \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}, \quad c_2 = \frac{1}{2} \sqrt{\mathbf{E}} \begin{bmatrix} -1 \\ -\sqrt{3} \end{bmatrix},
\]

and \( Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_2) \).

(a) Assuming the three hypotheses are equally likely, draw the optimal decision regions in the \((Y_1, Y_2)\) plane.

(b) Assume now that the a-priori probabilities for the hypotheses are given by \( \Pr\{H = 0\} = \frac{1}{2}, \quad \Pr\{H = 1\} = \Pr\{H = 2\} = \frac{1}{4} \). Draw the decision regions in the \((L_1, L_2)\) plane where

\[
L_i := \frac{f_{Y|H}(Y|i)}{f_{Y|H}(Y|0)}, \quad i = 1, 2.
\]

PROBLEM 2. \((Q\text{-}Function\ on\ Regions)\) Let \( X \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_2) \). For each of the three figures below, express the probability that \( X \) lies in the shaded region. You may use the \( Q \)-function when appropriate.

PROBLEM 3. \((16\text{-}PAM\ versus\ 16\text{-}QAM)\) The two signal constellations depicted below are used to communicate across an additive white Gaussian noise channel. Let the noise variance be \( \sigma^2 \). Each point represents a codeword \( c_i \) for some \( i \). Assume each codeword \( c_i \) is used with the same probability.
(a) For each signal constellation, compute the average probability of error $P_e$ as a function of the parameters $a$ and $b$, respectively.

(b) For each signal constellation, compute the average energy per symbol $\mathcal{E}$ as a function of the parameters $a$ and $b$, respectively:

$$\mathcal{E} = \sum_{i=1}^{16} P_H(i) \|c_i\|^2.$$

(c) Plot $P_e$ versus $\mathcal{E}/\sigma^2$ for both signal constellations and comment.

PROBLEM 4. (QPSK Decision Regions) Let $H \in \{0, 1, 2, 3\}$ and assume that when $H = i$ you transmit the codeword $c_i$ shown below. Under $H = i$, the receiver observes $Y = c_i + Z$.

(a) Draw the decoding regions assuming that $Z \sim \mathcal{N}(0, \sigma^2 I_2)$ and that $P_H(i) = 1/4$, $i \in \{0, 1, 2, 3\}$.

(b) Draw the decoding regions (qualitatively) assuming $Z \sim \mathcal{N}(0, \sigma^2 I_2)$ and $P_H(0) = P_H(2) > P_H(1) = P_H(3)$. Justify your answer.
(c) Assume again that \( P_H(i) = 1/4, \ i \in \{0,1,2,3\} \) and that \( Z \sim \mathcal{N}(0,K) \), where 
\[
K = \begin{pmatrix}
\sigma^2 & 0 \\
0 & 4\sigma^2
\end{pmatrix}.
\]
How do you decode now?

PROBLEM 5. (QAM with Erasure) Consider a QAM receiver that outputs a special symbol \( \delta \) (called erasure) whenever the observation falls in the shaded area shown in the figure below. Assume that \( c_0 \in \mathbb{R}^2 \) is transmitted and that \( Y = c_0 + N \) is received where \( N \sim \mathcal{N}(0,\sigma^2I_2) \). Let \( P_{0i}, \ i = 0,1,2,3 \) be the probability that the receiver outputs \( \hat{H} = i \) and let \( P_{0\delta} \) be the probability that it outputs \( \delta \). Determine \( P_{00}, P_{01}, P_{02}, P_{03} \) and \( P_{0\delta} \).

![Diagram of QAM with Erasure]

REMARK. If we choose \( b - a \) large enough, we can make sure that the probability of the error is very small (we say that an error occurred if \( \hat{H} = i, \ i \in \{0,1,2,3\} \) and \( H \neq \hat{H} \)). When \( \hat{H} = \delta \), the receiver can ask for a retransmission of \( H \). This requires a feedback channel from the receiver to the sender. In most practical applications, such a feedback channel is available.

PROBLEM 6. (Antenna Array) The following problem relates to the design of multi-antenna systems. Consider the binary equiprobable hypothesis testing problem:

\[
\begin{align*}
H = 0 & : \ Y_1 = A + Z_1, \ Y_2 = A + Z_2 \\
H = 1 & : \ Y_1 = -A + Z_1, \ Y_2 = -A + Z_2,
\end{align*}
\]

where \( Z_1, Z_2 \) are independent Gaussian random variables with different variances \( \sigma^2_1 \neq \sigma^2_2 \), that is, \( Z_1 \sim \mathcal{N}(0,\sigma^2_1) \) and \( Z_2 \sim \mathcal{N}(0,\sigma^2_2) \). \( A > 0 \) is a constant.

(a) Show that the decision rule that minimizes the probability of error (based on the observable \( Y_1 \) and \( Y_2 \)) can be stated as

\[
\sigma^2_2 y_1 + \sigma^2_1 y_2 \overset{?}{=} 0.
\]

(b) Draw the decision regions in the \((Y_1, Y_2)\) plane for the special case where \( \sigma_1 = 2\sigma_2 \).

(c) Evaluate the probability of the error for the optimal detector as a function of \( \sigma^2_1, \sigma^2_2 \) and \( A \).