ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 5 Principles of Digital Communications

Solutions to Problem Set 2 Mar. 3, 2015

SOLUTION 1.

(a) Let $l(y)$ be the number of 0's in the sequence y.

$$P_{Y|H}(y|0) = \frac{1}{2^{2k}}$$

$$P_{Y|H}(y|1) = \begin{cases} \frac{1}{\binom{2k}{k}}, & \text{if } l = k \\ 0, & \text{otherwise} \end{cases}$$

(b) The ML decision rule is:

$$P_{Y|H}(y|1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} P_{Y|H}(y|0).$$

Because $\frac{1}{\binom{2k}{k}} > \frac{1}{2^{2k}}$ for any value of k, the ML decision rule becomes

$$\hat{H} = \begin{cases} 0, & \text{if } l(y) \neq k \\ 1, & \text{if } l(y) = k. \end{cases}$$

The single number needed is $l(y)$, the number of 0's in the sequence y.

(c) The decision rule that minimizes error probability is the MAP rule:

$$P_{Y|H}(y|1)P_H(1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} P_{Y|H}(y|0)P_H(0).$$

The MAP decision rule gives $\hat{H} = 0$ whenever $l(y) \neq k$. When $l(y) = k$:

$$\hat{H} = \begin{cases} 0, & \text{if } \frac{\binom{2k}{k}}{2^{2k}} \geq \frac{P_H(1)}{P_H(0)} \\ 1, & \text{otherwise.} \end{cases}$$

(d) *Trivial solution:* If $P_H(1) = 1$ then $\hat{H} = 1$ for all y (In this case, $l(y) = k$ is guaranteed). Similarly, if $P_H(0) = 1$ then $\hat{H} = 0$ for all y.

Now assume $P_H(1) \neq 1$. Then there is a nonzero probability that $l(y) \neq k$, in which case $\hat{H} = 0$. The MAP decision rule always chooses $\hat{H} = 0$ if

$$\frac{\binom{2k}{k}}{2^{2k}} \geq \frac{P_H(1)}{P_H(0)} \Longleftrightarrow P_H(0) \geq \frac{\frac{1}{\binom{2k}{k}}}{\frac{1}{\binom{2k}{k}} + \frac{1}{2^{2k}}}.$$

Solution 2.

(a) A and B must be chosen such that the suggested functions become valid probability density functions, i.e. $\int_0^1 f_{Y|H}(y|i)dy = 1$ for $i = 0, 1$. This yields $A = 4/3$ and $B = 6/7$. (A quicker way is to draw the functions and find the area by looking at the drawings.)

(b) Let us first find the marginal of Y, i.e.

$$f_Y(y) = f_{Y|H}(y|0)P_H(0) + f_{Y|H}(y|1)P_H(1) = C - Dy,$$

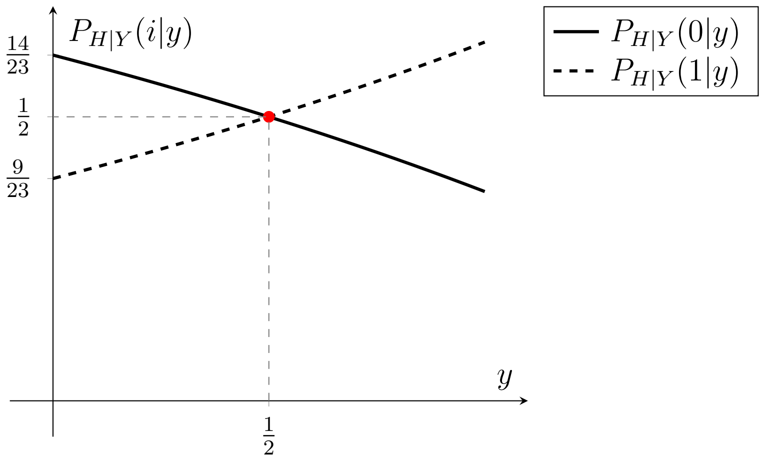
where we find $C = 23/21$ and $D = 4/21$. Then, applying Bayes' rule gives

$$P_{H|Y}(0|y) = \frac{f_{Y|H}(y|0)P_H(0)}{f_Y(y)} = \frac{1}{2}\frac{A - \frac{A}{2}y}{C - Dy} = \frac{1}{2}\frac{4/3 - 2/3y}{23/21 - 4/21y},$$

and similarly

$$P_{H|Y}(1|y) = \frac{f_{Y|H}(y|1)P_H(1)}{f_Y(y)} = \frac{1}{2}\frac{B + \frac{B}{3}y}{C - Dy} = \frac{1}{2}\frac{6/7 + 2/7y}{23/21 - 4/21y}.$$

(c) Here is a plot of $P_{H|Y}(0|y)$ and $P_{H|Y}(1|y)$:



The threshold is where the two a posteriori probabilities are equal,

$$\frac{1}{2}\frac{4/3 - 2/3y}{23/21 - 4/21y} = \frac{1}{2}\frac{6/7 + 2/7y}{23/21 - 4/21y},$$

or equivalently,

$$4/3 - 2/3y = 6/7 + 2/7y.$$

The y that satisfies this equation is our threshold θ, thus $\theta = \frac{1}{2}$.

(d) The probability that we decide $\hat{H}_\gamma(y) = 1$ when in reality $H = 0$ is just the probability that y is *larger* than the threshold given that $H = 0$, which is

$$\begin{aligned}
\Pr\{Y > \gamma | H = 0\} &= \int_\gamma^1 f_{Y|H}(y|0)dy = \int_\gamma^1 \left(A - \frac{A}{2}y\right) dy \\
&= A(1-\gamma) - \frac{A}{2}\frac{1-\gamma^2}{2} \\
&= \frac{4(1-\gamma)}{3} - \frac{1-\gamma^2}{3}.
\end{aligned}$$

(e) By analogy to the previous question,

$$\begin{aligned}
\Pr\{Y < \gamma | H = 1\} &= \int_0^\gamma f_{Y|H}(y|1)dy = \int_0^\gamma \left(B + \frac{B}{3}y\right) dy \\
&= B\gamma + \frac{B}{3}\frac{\gamma^2}{2} \\
&= \frac{6\gamma}{7} + \frac{\gamma^2}{7}.
\end{aligned}$$

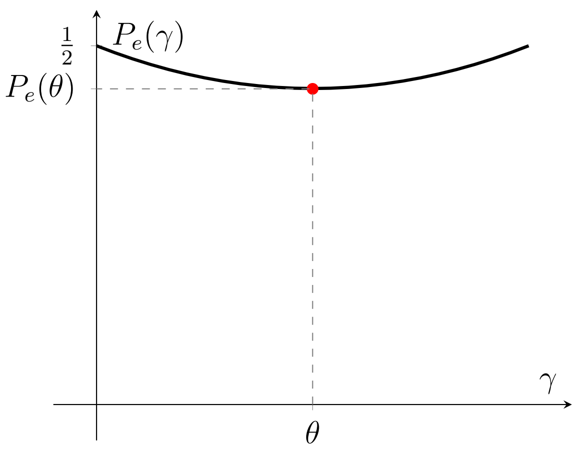
$$\begin{aligned}
P_e(\gamma) &= \Pr\{Y > \gamma | H = 0\} P_H(0) + \Pr\{Y < \gamma | H = 1\} P_H(1) \\
&= \frac{1}{2}\left(\frac{4(1-\gamma)}{3} - \frac{1-\gamma^2}{3} + \frac{6\gamma}{7} + \frac{\gamma^2}{7}\right)
\end{aligned}$$

For $\gamma = \theta = 0.5$, we find $P_e(\theta) = 0.44$.

(f) To minimize P_e over γ, we observe that $P_e(\gamma)$ is a convex function of γ (it is a parabola with positive coefficient of γ^2), hence we may take the derivative with respect to γ and set it equal to zero, i.e.

$$\frac{d}{d\gamma}P_e(\gamma) = \frac{1}{2}\left(-\frac{4}{3} + \frac{2\gamma}{3} + \frac{6}{7} + \frac{2\gamma}{7}\right)$$

Setting this equal to zero, we find $\gamma = 0.5$. We observe that the value of γ which minimizes $P_e(\gamma)$ is equal to θ. This was expected, because the MAP decision rule minimizes the error probability.



Solution 3.

Remark (An explanation regarding the title of this problem). *Independent and identically distributed means that all $Y_1, \ldots, Y_k$ have the same probability mass function and are independent of each other.* First-order Markov *means that $Y_1, \ldots, Y_k$ depend on each other in a particular way: the probability mass function Y_i depends on the value of Y_{i-1}, but given the value of Y_{i-1}, it is independent of $Y_1, \ldots, Y_{i-2}$. Thus, in this problem, we observe a binary sequence, and we want to know whether it has been generated by an* i.i.d. *(independent and identically distributed) source or by a first-order Markov source.*

(a) We first know that

$$P_{Y|H}(y|0) = \left(\frac{1}{2}\right)^k \qquad \forall y \in \{0,1\}^k$$

and

$$P_{Y|H}(y|1) = \frac{1}{2}\left(\frac{1}{4}\right)^l \left(\frac{3}{4}\right)^{k-l-1},$$

where l is the number of times the observed sequence $y \in \{0,1\}^k$ changes from zero to one or one to zero, i.e. the number of transitions in the observed sequence.

Since the two hypotheses are equally likely, we find

$$\frac{P_{Y|H}(y|1)}{P_{Y|H}(y|0)} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{P_H(0)}{P_H(1)} = 1.$$

Plugging in, we obtain

$$\frac{1/2 \cdot (1/4)^l \cdot (3/4)^{k-l-1}}{(1/2)^k} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 1.$$

(b) The sufficient statistic here is simply the number of transitions l; this entirely specifies the likelihood ratio.

Solution 4. Note that since noise samples are i.i.d., the conditional probability distribution functions under H_0 and H_1 will respectively be

$$f_{Y|H}(y|0) = \prod_{k=1}^{n} f_Z(y_k)$$
$$f_{Y|H}(y|1) = \prod_{k=1}^{n} f_Z(y_k - 2A)$$

where $f_Z(z)$ is the pdf of Z_k, $k = 1, \ldots, n$. Furthermore, since the two hypotheses are equi-probable, the MAP decision reduces to the ML decision rule.

(a) Plugging the pdf of Z the MAP decision rule becomes

$$\frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{k=1}^n (y_k-2A)^2} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{k=1}^n y_k^2}.$$

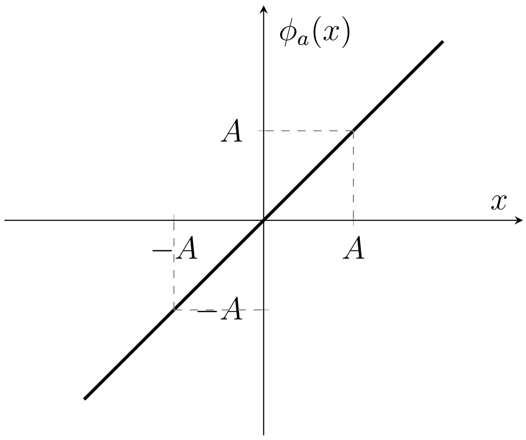
Simplifying the common factor $\frac{1}{(2\pi\sigma^2)^{n/2}}$ and taking the logarithm we have

$$-\frac{1}{2\sigma^2}\sum_{k=1}^n (y_k-2A)^2 \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} -\frac{1}{2\sigma^2}\sum_{k=1}^n y_k^2.$$

Further simplifications reduce the MAP decision rule to

$$\sum_{k=1}^n y_k \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} nA \quad \Longleftrightarrow \quad \sum_{k=1}^n (y_k - A) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 0.$$

Hence $\phi_a(x) = x$.



(b) Similarly, the MAP decision rule is now

$$\frac{1}{(2\sigma^2)^{n/2}} e^{-\frac{\sqrt{2}}{\sigma}\sum_{k=1}^n |y_k-2A|} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{1}{(2\sigma^2)^{n/2}} e^{-\frac{\sqrt{2}}{\sigma}\sum_{k=1}^n |y_k|}.$$

Simplifying common terms and taking the logarithm gives

$$-\frac{\sqrt{2}}{\sigma}\sum_{k=1}^n |y_k-2A| \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} -\frac{\sqrt{2}}{\sigma}\sum_{k=1}^n |y_k|.$$

We can write the above in the desired form by noting that

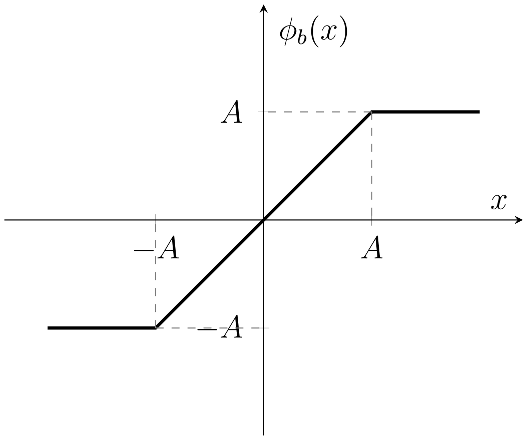
$$|x| - |x-2A| = 2\phi_b(x-A)$$

where

$$\phi_b(x) \triangleq \begin{cases} A & \text{if } x \geq A, \\ x & \text{if } -A \leq x \leq A, \\ -A & \text{if } x \leq -A. \end{cases}$$

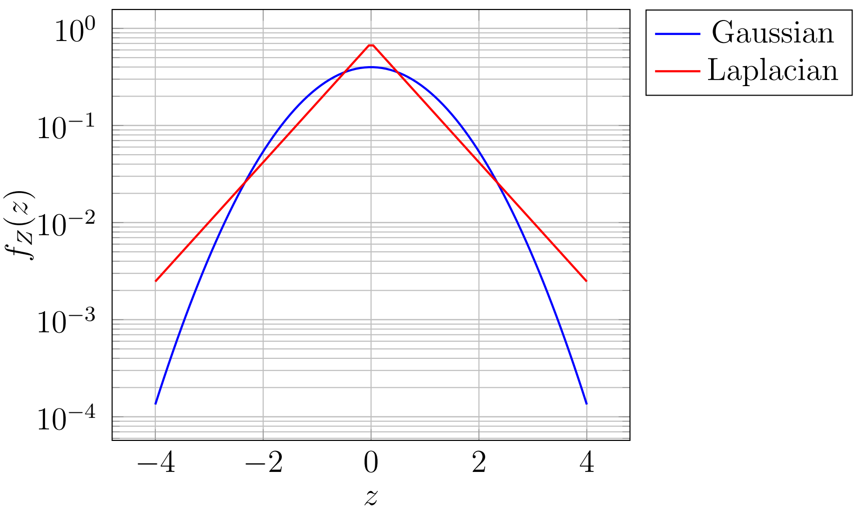
Thus the MAP decision rule will be

$$\sum_{k=1}^{n} \phi_b(y_k - A) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 0.$$



Again note that only the value of A is needed for implementing the decision rule.

Here you see a plot of two noise distributions for $\sigma = 1$:



The Laplacian distribution has larger 'tails'; it puts more mass on very large positive and very large (in absolute value) negative values of z. Because of this, for the decision in part (b) the optimal choice is to first "clip" the input data y_k, $k = 1, \ldots, n$ so that these high values do not influence the decision.

Solution 5. Repeating the same steps as in the previous exercise, we see that the MAP decision rule is

$$\frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{n}(y_k-A)^2} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{n}(y_k+A)^2}$$

Simplifying the common positive factor of $\frac{1}{(2\pi\sigma^2)^{n/2}}$ and taking the logarithm we have

$$-\frac{1}{2\sigma^2}\sum_{k=1}^{n}(y_k-A)^2 \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} -\frac{1}{2\sigma^2}\sum_{k=1}^{n}(y_k+A)^2.$$

which can further be simplified to

$$\sum_{k=1}^{n} y_k \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 0.$$

Note that for implementing the decision rule the receiver does not need to know the exact value of A whereas in the previous problem A was a required parameter.

Solution 6.

(a) We have a binary hypothesis testing problem, here the hypothesis H is the answer you will select, and your decision will be based on the observation of $\hat{H}_L$ and $\hat{H}_R$. Let H take value 1 if answer 1 is chosen, and value 2 if answer 2 is chosen. In this case, we can write the MAP decision rule as follows:

$$\Pr\left\{H=1|\hat{H}_L=1,\hat{H}_R=2\right\} \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}} \Pr\left\{H=2|\hat{H}_L=1,\hat{H}_R=2\right\}.$$

From the problem setting we know the priors $\Pr\{H=1\}$ and $\Pr\{H=2\}$; we can also determine the conditional probabilities $\Pr\left\{\hat{H}_L=1|H=1\right\}$, $\Pr\left\{\hat{H}_L=1|H=2\right\}$, $\Pr\left\{\hat{H}_R=2|H=1\right\}$ and $\Pr\left\{\hat{H}_R=2|H=2\right\}$ (we have $\Pr\left\{\hat{H}_L=1|H=1\right\}=0.9$ and $\Pr\left\{\hat{H}_L=1|H=2\right\}=0.1$). Introducing these quantities and using the Bayes rule we can formulate the MAP decision rule as

$$\frac{\Pr\left\{\hat{H}_L=1,\hat{H}_R=2|H=1\right\}\Pr\{H=1\}}{\Pr\left\{\hat{H}_L=1,\hat{H}_R=2\right\}} \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}} \frac{\Pr\left\{\hat{H}_L=1,\hat{H}_R=2|H=2\right\}\Pr\{H=2\}}{\Pr\left\{\hat{H}_L=1,\hat{H}_R=2\right\}}.$$

Now, assuming that the event $\{\hat{H}_L=1\}$ is independent of the event $\{\hat{H}_R=2\}$ and simplifying the expression, we obtain

$$\Pr\left\{\hat{H}_L=1|H=1\right\}\Pr\left\{\hat{H}_R=2|H=1\right\}\Pr\{H=1\} \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}}$$
$$\Pr\left\{\hat{H}_L=1|H=2\right\}\Pr\left\{\hat{H}_R=2|H=2\right\}\Pr\{H=2\},$$

which is our final decision rule.

(b) Evaluating the preceding decision rule, we have

$$0.9 \cdot 0.3 \cdot 0.25 \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}} 0.1 \cdot 0.7 \cdot 0.75,$$

which gives

$$0.0675 \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}} 0.0525.$$

This implies that the answer $\hat{H}$ is equal to 1.