ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 4 Principles of Digital Communications
Problem Set 2 Feb. 25, 2015

PROBLEM 1. *(Hypothesis Testing: Uniform and Uniform)* Consider a binary hypothesis testing problem in which the hypotheses $H = 0$ and $H = 1$ occur with probability $P_H(0)$ and $P_H(1) = 1 - P_H(0)$, respectively. The observable Y takes values in $\{0, 1\}^{2k}$, where k is a fixed positive integer. When $H = 0$, each component of Y is 0 or a 1 with probability $\frac{1}{2}$ and components are independent. When $H = 1$, Y is chosen uniformly at random from the set of all sequences of length $2k$ that have an equal number of ones and zeros. There are $\binom{2k}{k}$ such sequences.

(a) What is $P_{Y|H}(y|0)$? What is $P_{Y|H}(y|1)$?

(b) Find a maximum likelihood decision rule for H based on y. What is the single number you need to know about y to implement this decision rule?

(c) Find a decision rule that minimizes the error probability.

(d) Are there values of $P_H(0)$ such that the decision rule that minimizes the error probability always choses the same hypotheses regardless of y? If yes, what are these values, and what is the decision?

PROBLEM 2. *(The "Wetterfrosch")* Let us assume that a "weather frog" bases his forecast of tomorrow's weather entirely on today's air pressure. Determining a weather forecast is a hypothesis testing problem. For simplicity, let us assume that the weather frog only needs to tell us if the forecast for tomorrow's weather is "sunshine" or "rain". Hence we are dealing with binary hypothesis testing. Let $H = 0$ mean "sunshine" and $H = 1$ mean "rain". We will assume that both values of H are equally likely, i.e. $P_H(0) = P_H(1) = \frac{1}{2}$. Measurements over several years have led the weather frog to conclude that on a day that precedes sunshine the pressure may be modeled as a random variable Y with the following probability density function:

$$f_{Y|H}(y|0) = \begin{cases} A - \frac{A}{2}y, & 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, the pressure on a day that precedes a rainy day is distributed according to

$$f_{Y|H}(y|1) = \begin{cases} B + \frac{B}{3}y, & 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

The weather frog's purpose in life is to guess the value of H after measuring Y.

(a) Determine A and B.

(b) Find the *a posteriori probability* $P_{H|Y}(0|y)$. Also find $P_{H|Y}(1|y)$.

(c) Plot $P_{H|Y}(0|y)$ and $P_{H|Y}(1|y)$ as a function of y. Show that the implementation of the decision rule $\hat{H}(y) = \arg\max_i P_{H|Y}(i|y)$ reduces to

$$\hat{H}_\theta(y) = \begin{cases} 0, & \text{if } y \le \theta \\ 1, & \text{otherwise,} \end{cases}$$

for some threshold θ and specify the threshold's value.

(d) Now assume that you implement the decision rule $\hat{H}_\gamma(y)$ for some arbitrary $\gamma \in \mathbb{R}$. Determine, as a function of γ, the probability that the decision rule decides $\hat{H} = 1$ given that $H = 0$. This probability is denoted $\Pr\left\{\hat{H}(Y) = 1 | H = 0\right\}$.

(e) For the same decision rule, determine the probability of error $P_e(\gamma)$ as a function of γ. Evaluate your expression at $\gamma = \theta$.

(f) Find the γ that minimizes $P_e(\gamma)$ and compare your result to θ. Could you have found the minimizing γ without any calculation?

Problem 3. *(Independent and Identically Distributed versus First-Order Markov)* Consider testing two equally likely hypotheses $H = 0$ and $H = 1$. The observable $Y = (Y_1, \dots, Y_k)^\mathsf{T}$ is a k-dimensional binary vector. Under $H = 0$ the components of the vector Y are independent uniform random variables (also called Bernoulli$(1/2)$ random variables). Under $H = 1$, the component Y_1 is also uniform, but the components Y_i, $2 \le i \le k$, are distributed as follows:

$$P_{Y_i|Y_1,\dots,Y_{i-1}}(y_i|y_1,\dots,y_{i-1}) = \begin{cases} 3/4, & \text{if } y_i = y_{i-1} \\ 1/4, & \text{otherwise.} \end{cases} \quad (1)$$

(a) Find the decision rule that minimizes the probability of error. *Hint:* Write down a short sample sequence $(y_1, \dots, y_k)$ and determine its probability under each hypothesis. Then generalize.

(b) Give a simple sufficient statistic for this decision. (For the purpose of this question, a sufficient statistic is a function of y with the property that a decoder that observes y can not achieve a smaller error probability than a MAP decoder that observes this function of y.)

Problem 4. *(Gaussian versus Laplacian Noise)* Consider the following binary hypothesis testing problem. The hypotheses are equally likely and the observable $Y = (Y_1, \dots, Y_n)^\mathsf{T}$ is a n-dimensional real vector whose components are:

$$H_0 : Y_k = Z_k \qquad \text{versus} \qquad H_1 : Y_k = 2A + Z_k, \quad k = 1, \dots, n,$$

where $A > 0$ is a positive constant and $Z_1, \dots, Z_n$ is an i.i.d. noise sequence.

In each of following cases, show that the MAP decision rule reduces to

$$\hat{H}(y) = \begin{cases} 0 & \text{if } \sum_{k=1}^{n} \phi(y - A) < 0, \\ 1 & \text{otherwise.} \end{cases}$$

and find the function $\phi(x)$.

(a) If Z_k are i.i.d. Gaussian noise samples with zero mean and variance σ^2, i.e.

$$f_{Z_k}(z_k) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{z_k^2}{2\sigma^2}}, \quad k = 1, \dots, n.$$

(b) If Z_k are i.i.d Laplacian noise samples with variance σ^2. That is,

$$f_{Z_k}(z_k) = \frac{1}{\sigma\sqrt{2}} e^{-\frac{\sqrt{2}}{\sigma}|z_k|}, \quad k = 1, \dots, n.$$

Plot the noise pdfs of part (a) and (b) for some fixed value of σ (take $\sigma = 1$ for convenience). Compare the distributions and the functions $\phi(x)$ that you have found. Can you find an intuitive explanation for their difference?

Problem 5. Consider the binary hypothesis testing problem where the hypotheses are equally likely and the observable $Y = (Y_1, \dots, Y_n)^\mathsf{T}$ is a n-dimensional real vector with components defined as

$$H_0 : Y_k = -A + Z_k \qquad \text{versus} \qquad H_1 : Y_k = A + Z_k, \quad k = 1, \dots, n,$$

where $A > 0$ is a positive constant and $Z_1, \dots, Z_n$ are i.i.d. Gaussian noise samples with variance σ^2. Find the decision rule that minimizes the probability of error. Compare your answer with that of Problem 4 part (a). What can you conclude?

Problem 6. *(Multiple Choice Exam)* You are taking a multiple choice exam. Question number 5 allows for two possible answers. According to your first impression, answer 1 is correct with probability $1/4$ and answer 2 is correct with probability $3/4$.

You would like to maximize your chance of giving the correct answer and you decide to have a look at what your left and right neighbors have to say.

The left neighbor has answered $\hat{H}_L = 1$. He is an excellent student who has a record of being correct 90% of the time.

The right neighbor has answered $\hat{H}_R = 2$. He is a weaker student who is correct 70% of the time.

(a) You decide to use your first impression as a prior and to consider $\hat{H}_L$ and $\hat{H}_R$ as observations. Describe the corresponding hypothesis testing problem.

(b) What is your answer $\hat{H}$? Justify it.