

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 3

Principles of Digital Communications

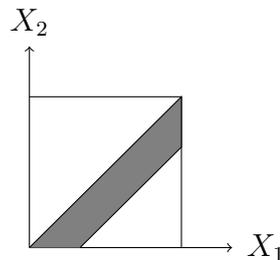
Solutions to Problem Set 1

Feb. 20, 2015

SOLUTION 1. In each case, the shaded region represents the (X_1, X_2) values satisfying the corresponding inequalities. Since X_1 and X_2 are independent and uniformly distributed, the area of the shaded region gives the probability of the inequality being satisfied. We use $\Pr\{\cdot\}$ to denote the probability of an event.

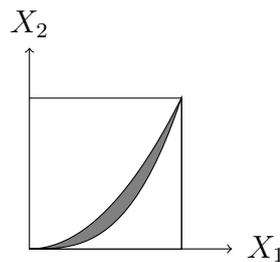
(a)

$$\Pr\left\{0 \leq X_1 - X_2 \leq \frac{1}{3}\right\} = \frac{1}{2} - \frac{1}{2} \times \left(\frac{2}{3} \times \frac{2}{3}\right) = \frac{5}{18}.$$



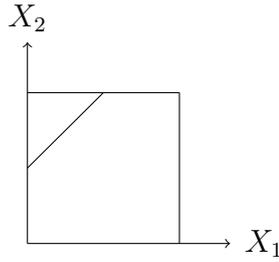
(b)

$$\Pr\{X_1^3 \leq X_2 \leq X_1^2\} = \int_0^1 (x^2 - x^3) dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12}.$$



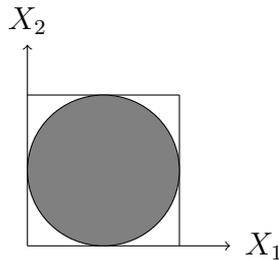
(c)

$$\Pr\left\{X_2 - X_1 = \frac{1}{2}\right\} = 0.$$



(d)

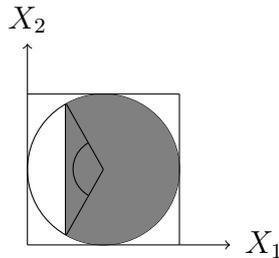
$$\Pr \left\{ \left(X_1 - \frac{1}{2} \right)^2 + \left(X_2 - \frac{1}{2} \right)^2 \leq \left(\frac{1}{2} \right)^2 \right\} = \pi \left(\frac{1}{2} \right)^2 = \frac{\pi}{4}.$$



(e) In this part we have

$$\begin{aligned} & \Pr \left\{ \left(X_1 - \frac{1}{2} \right)^2 + \left(X_2 - \frac{1}{2} \right)^2 \leq \left(\frac{1}{2} \right)^2 \mid X_1 \geq \frac{1}{4} \right\} \\ &= \frac{\Pr \left\{ \left(X_1 - \frac{1}{2} \right)^2 + \left(X_2 - \frac{1}{2} \right)^2 \leq \left(\frac{1}{2} \right)^2, X_1 \geq \frac{1}{4} \right\}}{\Pr \left\{ X_1 \geq \frac{1}{4} \right\}} \\ &= \frac{\frac{\pi}{6} + \frac{\sqrt{3}}{16}}{\frac{3}{4}} \end{aligned}$$

It can easily be seen that probability term in the numerator is equal to the area of the shaded region in the figure below. We can divide the shaded area to two parts, triangular and sub circular. It is easy to show that the angle of the triangle on the picture is 120° so the sub circular part consists of $\frac{2}{3}$ of the circle area. So sub circular part area is $\frac{2}{3} \pi \left(\frac{1}{2} \right)^2 = \frac{\pi}{6}$ and triangular part area is $\frac{\sqrt{3}}{16}$ and summing the area of these two parts, we reach the final result.



SOLUTION 2. (a) First, we find the probability of the complement of the event, namely the probability of drawing only black balls. This probability is equal to

$$\Pr \{ \text{All } k \text{ balls are black} \} = \frac{\binom{n}{k}}{\binom{m+n}{k}}.$$

Therefore the probability of drawing at least one white ball is equal to

$$\Pr \{ \text{At least one ball is white} \} = 1 - \frac{\binom{n}{k}}{\binom{m+n}{k}}.$$

(b) Define the following random variables

$$X = \begin{cases} 0 & \text{If the chosen coin is fair,} \\ 1 & \text{otherwise,} \end{cases}$$

and

$$Y = \begin{cases} 00 & \text{If both outcomes are tail,} \\ 01 & \text{If the first one is tail, the second one is head,} \\ 10 & \text{If the first one is head, the second one is tail,} \\ 11 & \text{If both outcomes are head.} \end{cases}$$

Having defined these random variables, we want to compute $\Pr \{ X = 0 | Y = 11 \}$. So we can write

$$\begin{aligned} \Pr \{ X = 0 | Y = 11 \} &= \frac{\Pr \{ Y = 11 | X = 0 \} \Pr \{ X = 0 \}}{\Pr \{ Y = 11 \}} \\ &= \frac{1/2 \times 1/4}{\Pr \{ Y = 11 \}} \\ &= \frac{1/8}{\Pr \{ Y = 11 \}}. \end{aligned}$$

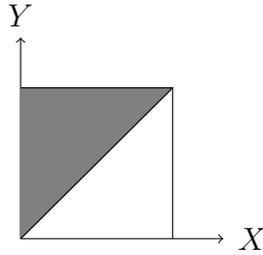
Then for $\Pr \{ Y = 11 \}$ we have

$$\begin{aligned} \Pr \{ Y = 11 \} &= \Pr \{ X = 0 \} \cdot \Pr \{ Y = 11 | X = 0 \} + \Pr \{ X = 1 \} \cdot \Pr \{ Y = 11 | X = 1 \} \\ &= 1/2 \times 1/4 + 1/2 \times 1 \\ &= 5/8. \end{aligned}$$

So, finally we have

$$\Pr \{ X = 0 | Y = 11 \} = \frac{1/8}{5/8} = \frac{1}{5}.$$

SOLUTION 3. The probability mass has been distributed uniformly on the upper triangular area according to the shape below:

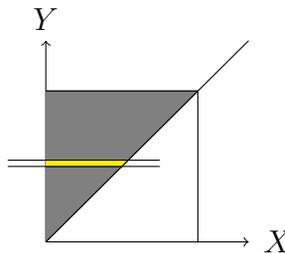


- (a) If X and Y were independent then the distribution of X would not depend on Y . This is clearly not the case. In fact, the range of values taken by X is between 0 and Y .
- (b) The integral of $f_{X,Y}(x,y)$ must be 1. Hence $A \times \frac{1}{2} = 1$ and so $A = 2$.
- (c) We know that $f_Y(y) dy = \Pr \{y < Y < y + dy\}$ but for a special y as can be seen from the figure below this probability mass is equal to A times the area of a rectangle with length y and width dy when $0 \leq y \leq 1$.

$$f_Y(y) = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Or more formally

$$f_Y(y) = \int_0^1 f_{X,Y}(x,y) dx = \int_0^y 2 dx = 2y$$



- (d) Under the condition $Y = y$, the random variable X is uniformly distributed between 0 and y and so $f(y) = \mathbb{E}[X|Y = y] = \frac{y}{2}$.
- (e) $f(Y)$ is a function of Y so it is a random variable and we can compute its expected value.

$$\mathbb{E}[f(Y)] = \int_0^1 f(y)f_Y(y) dy = \int_0^1 y^2 dy = \frac{1}{3}$$

- (f) We compute the $\mathbb{E}[X]$ using the definition.

$$\mathbb{E}[X] = \iint x f_{X,Y}(x,y) dx dy = \int_0^1 \left[\int_0^y 2x dx \right] dy = \frac{1}{3},$$

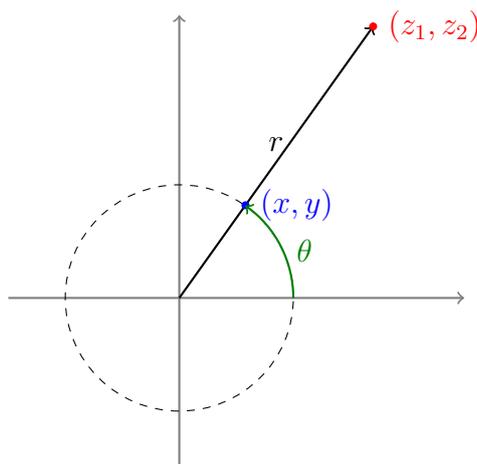
and it is seen that $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$. This result, which holds in general, is named the law of total expectation.

SOLUTION 4. It is easy to check $X^2 + Y^2 = 1$ hence (X, Y) is a point on the unit circle.

As Z_1 and Z_2 are independent Gaussians, the their joint probability density is

$$f_{Z_1, Z_2}(z_1, z_2) = f_{Z_1}(z_1)f_{Z_2}(z_2) = \frac{1}{\sigma^2 2\pi} e^{-\frac{z_1^2 + z_2^2}{2\sigma^2}}$$

Note that this joint density depends on (z_1, z_2) only through its distance from the origin $r = \sqrt{z_1^2 + z_2^2}$, and not on the angle θ . Consequently, the angle Θ is uniformly distributed. Since the (X, Y) is a scaled version of (Z_1, Z_2) without changing the angle, it is uniformly distributed on the circle (see the figure below).



A more formal justification of the fact that the angle is uniformly distributed is as follows. Let $R := \sqrt{Z_1^2 + Z_2^2}$ and $\Theta := \tan^{-1}(Z_2/Z_1)$ be the *polar* coordinates of the point (Z_1, Z_2) . We can conclude that the joint density $f_{R, \Theta}(r, \theta)$ is only a function of r . Consequently,

$$f_{\Theta}(\theta) = \int_0^{\infty} f_{R, \Theta}(r, \theta) dr = C \quad \text{for } \forall \theta \in [0, 2\pi),$$

where C is a *constant* and (as $\int_0^{2\pi} f_{\Theta}(\theta) d\theta = 1$) is equal to $\frac{1}{2\pi}$. Thus, Θ has a uniform distribution in $[0, 2\pi)$.

REMARK 1. One can check that the random variable R has a Rayleigh distribution,

$$f_R(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}.$$

Furthermore the random variables R and Θ are also independent.

SOLUTION 5.

Recall that:

- By definition, X and Y are uncorrelated if and only if

$$0 = \text{cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Hence $\text{cov}(X, Y) = 0$ is equivalent to the condition $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

- X and Y are independent when $f_{XY} = f_X f_Y$.

- (a) Assume that the random variables X and Y are independent. Then:

$$\begin{aligned} \mathbb{E}[XY] &= \iint xy f_{X,Y}(x, y) \, dx \, dy = \iint xy f_X(x) f_Y(y) \, dx \, dy \\ &= \int x f_X(x) \, dx \int y f_Y(y) \, dy = \mathbb{E}[X]\mathbb{E}[Y] \end{aligned}$$

where the second equality follows from the assumption that X and Y are independent. Hence, if X and Y are independent, they are also uncorrelated.

- (b) X and Y are obviously dependent. For example, $X = 0$ implies $U = 0$ and $V = 0$. Hence it implies also $Y = 0$. The marginals of X and Y are:

$$X = \begin{cases} 0, & \text{with prob. } \frac{1}{4} \\ 1, & \text{with prob. } \frac{1}{2} \\ 2, & \text{with prob. } \frac{1}{4}, \end{cases}$$

$$Y = \begin{cases} 0, & \text{with prob. } \frac{1}{2} \\ 1, & \text{with prob. } \frac{1}{2}. \end{cases}$$

The mean for X is $\mathbb{E}[X] = 1$ and for Y it is $\mathbb{E}[Y] = \frac{1}{2}$. Finally, we have that

$$\mathbb{E}[XY] = \frac{1}{4} \times 0 \times 0 + \frac{1}{4} \times 1 \times 1 + \frac{1}{4} \times 1 \times 1 + \frac{1}{4} \times 0 \times 2 = \frac{1}{2}$$

From these two:

$$\text{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0,$$

thus, X and Y are uncorrelated, even though they are dependent.

SOLUTION 6. The unit sphere is defined by equation $x^2 + y^2 + z^2 = 1$. Hence $X^2 + Y^2 + Z^2 = 1$ with probability 1 which implies $\mathbb{E}[X^2 + Y^2 + Z^2] = 1$.

By linearity of expectation,

$$\mathbb{E}[X^2 + Y^2 + Z^2] = \mathbb{E}[X^2] + \mathbb{E}[Y^2] + \mathbb{E}[Z^2] = 1. \quad (*)$$

Furthermore, because of symmetry X , Y and Z have the same marginal distributions. In particular, $\mathbb{E}[X^2] = \mathbb{E}[Y^2] = \mathbb{E}[Z^2]$. Using this in (*) we can conclude that $\mathbb{E}[X^2] = \frac{1}{3}$.